

HIGHER STRING FUNCTIONS, HIGHER-LEVEL APPELL FUNCTIONS, AND THE LOGARITHMIC $\widehat{sl}(2)_k/u(1)$ CFT MODEL

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ABSTRACT. We generalize the string functions $\mathcal{C}_{n,r}(\tau)$ associated with the coset $\widehat{sl}(2)_k/u(1)$ to higher string functions $\mathcal{A}_{n,r}(\tau)$ and $\mathcal{B}_{n,r}(\tau)$ associated with the coset $W(k)/u(1)$ of the W -algebra of the logarithmically extended $\widehat{sl}(2)_k$ conformal field model with positive integer k . The higher string functions occur in decomposing $W(k)$ characters with respect to level- k theta and Appell functions and their derivatives (the characters are neither quasiperiodic nor holomorphic, and therefore cannot decompose with respect to only theta-functions). The decomposition coefficients, to be considered “logarithmic parafermionic characters,” are given by $\mathcal{A}_{n,r}(\tau)$, $\mathcal{B}_{n,r}(\tau)$, $\mathcal{C}_{n,r}(\tau)$, and by the triplet $\mathcal{W}(p)$ -algebra characters of the $(p = k + 2, 1)$ logarithmic model. We study the properties of $\mathcal{A}_{n,r}$ and $\mathcal{B}_{n,r}$, which nontrivially generalize those of the classic string functions $\mathcal{C}_{n,r}$, and evaluate the modular group representation generated from $\mathcal{A}_{n,r}(\tau)$ and $\mathcal{B}_{n,r}(\tau)$; its structure inherits some features of modular transformations of the higher-level Appell functions and the associated transcendental function Φ .

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Там даже красовалась неясным логарифмом
 Абстрактная картина для общей красоты.
 Ю. Визбор

1. INTRODUCTION

The defining feature of logarithmic conformal field theories [3, 4, 5, 6], contrasting them from rational conformal field theories, is the presence of indecomposable representations of the chiral algebra. The interesting representation theory may be considered the basic reason underlying fascinating features of logarithmic conformal field models and their links with several related problems, e.g., in [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20]. In particular, modular group representations generated from characters in logarithmic models are of a different structure than the modular group representations occurring in rational models (cf. [21, 22, 23]).

In this paper, we derive a modular group representation of a “logarithmic” origin, generated from the *higher string functions* (for positive integer k and $1 \leq r \leq p = k + 2$, with $n - r \in 2\mathbb{Z} + 1$)

$$\mathcal{A}_{n,r}(q) = \frac{q^{-\frac{n^2}{4k}}}{\eta(q)^2} \sum_{a \in \mathbb{Z}} \sum_{j \geq 1} (-1)^{j+1} \left(a - \frac{n}{2k}\right)^2 \left(q^{\frac{1}{2}j(j-n) + \frac{(2ap+r)^2}{4p} + \frac{1}{2}j(2ap+r)} - (r \mapsto -r)\right),$$

$$\mathcal{B}_{n,r}(q) = \frac{q^{-\frac{n^2}{4k}}}{\eta(q)^2} \sum_{a \in \mathbb{Z}} \sum_{j \geq 1} (-1)^{j+1} \left(a - \frac{n}{2k}\right) \left(q^{\frac{1}{2}j(j-n) + \frac{(2ap+r)^2}{4p} + \frac{1}{2}j(2ap+r)} - (r \mapsto -r)\right),$$

which generalize the classic string functions [24, 25]

$$\mathcal{C}_{n,r}(q) = \frac{q^{-\frac{n^2}{4k}}}{\eta(q)^2} \sum_{a \in \mathbb{Z}} \sum_{j \geq 1} (-1)^{j+1} \left(q^{\frac{1}{2}j(j-n) + \frac{(2ap+r)^2}{4p} + \frac{1}{2}j(2ap+r)} - (r \mapsto -r)\right)$$

in an obvious way. That $\mathcal{A}_{n,r}$ and $\mathcal{B}_{n,r}$ can have reasonable modular properties is not obvious, however, and these properties are actually nontrivial. The most striking feature is that modular S -transformations of $\mathcal{A}_{n,r}$ and $\mathcal{B}_{n,r}$ involve the transcendental function

$$(1.1) \quad \Phi(\tau, \mu) = -\frac{i}{2\sqrt{-i\tau}} - \frac{1}{2} \int_{\mathbb{R}} dx e^{-\pi x^2} \frac{\sinh\left(\pi x \sqrt{-i\tau} \left(1 + 2\frac{\mu}{\tau}\right)\right)}{\sinh(\pi x \sqrt{-i\tau})}$$

introduced previously in studying $\widehat{s\ell}(2|1)$ characters [1]. Less striking but also interesting is that the modular transforms of $\mathcal{B}_{n,r}$ and $\mathcal{A}_{n,r}$ involve Φ and its derivative *times the characters* of the $(p, 1)$ logarithmic conformal field model. The underlying representation-theory reasons are briefly as follows.

We recall that the string functions $\mathcal{C}_{n,r}(q)$ are the coefficients in the decomposition of integrable $\widehat{s\ell}(2)_k$ characters with respect to level- k theta-functions. Their “logarithmic” generalizations $\mathcal{B}_{n,r}(q)$ and $\mathcal{A}_{n,r}(q)$ occur similarly in decomposing the characters of a

W -algebra $W(k)$ in a logarithmically extended minimal $\widehat{s\ell}(2)_k$ conformal field theory model [2]; they are thus associated with a logarithmic extension of $\widehat{s\ell}(2)_k/u(1)$. The modular transformations of $\mathcal{A}_{n,r}$ and $\mathcal{B}_{n,r}$ can then be found in much the same way as in the well-known case with $\mathcal{C}_{n,r}$. Both the technical details (page 4) and the result (page 5) make this undertaking interesting. But before describing these, we recall some motivation from logarithmic conformal field theory (we actually need only the characters and their modular transformations, and therefore some readers may well skip the next subsection).¹

1.1. Logarithmic conformal field theory background. The classic string functions $\mathcal{C}_{n,r}(q)$ are (modulo normalization) the characters of the coset $\widehat{s\ell}(2)_k/u(1)$ model — the parafermionic theory that could never complain about lack of attention since its appearance in [26] (e.g., see [27] and the references therein, [28, 29, 30, 31] in particular). The higher string functions $\mathcal{B}_{n,r}(q)$ and $\mathcal{A}_{n,r}(q)$ are “logarithmic extensions” of these characters in that they originate similarly to the $\mathcal{C}_{n,r}(q)$ from a logarithmically extended theory.

Logarithmic conformal field theories differ from rational ones in several ways, the two major effects being as follows. First, the chiral space of states of a logarithmic model is the sum not of all irreducible representations but of all indecomposable projective modules (cf. a discussion in [32, 2, 14]). Second, the chiral algebra itself extends to a larger, typically nonlinear W -algebra. Such extended algebras can be systematically identified as maximum local algebras acting in the kernel of the differential in certain complexes associated with screenings.

Logarithmic conformal models can be systematically defined by choosing a free-field realization, identifying the screenings that select the (nonextended, to begin with) chiral algebra as their centralizer, constructing a complex associated with the screenings, and then taking the kernel of the differential and the maximum local algebra acting there [22, 23, 32, 2].

When the nonextended symmetry is the Virasoro algebra, the chiral algebra is the triplet W -algebra $\mathcal{W}(p) = W_{2,3 \times (2p-1)}$ [33, 34] for $(p, 1)$ models or a triplet W -algebra [32] with generating currents of dimension $(2p-1)(2p'-1)$ for (p, p') models. For $(p, 1)$ models, in particular, the “screening-kernel” approach yields a “semi-explicit” construction [22, 23] of the currents generating the $\mathcal{W}(p)$ algebra (in terms of vertex operators and screenings; also see [35]) and a description of its $2p$ irreducible representations, whence their characters follow as (see [21] for their first derivation)

$$(1.2) \quad \psi_r^+(q) = \frac{r\theta_{r,p}(q) - 2\theta'_{r,p}(q)}{p\eta(q)}, \quad \psi_r^-(q) = \frac{r\theta_{p-r,p}(q) + 2\theta'_{p-r,p}(q)}{p\eta(q)}, \quad 1 \leq r \leq p.$$

¹A general context to which the results in this paper relate is that of mock theta-functions. That this particular “mockery” of theta functions has reasonable properties must be traceable to conformal field theory/representation theory reasons.

When the nonextended symmetry is $\widehat{s\ell}(2)_k$ with positive integer k , the currents generating the corresponding extended W -algebra $W(k)$ are of dimension $4p - 2$ (and charge $\pm(2p - 1)$), $p = k + 2$ [2]. But the “screening-kernel” approach suffers from a mismatch between the number of screenings (two) selecting the $\widehat{s\ell}(2)$ algebra as their centralizer and the number of free fields (three) entering the free-field construction of $\widehat{s\ell}(2)$ (a “runaway” direction in the 3-space of vertex-operator momenta is associated with the spectral flow). These two numbers may be equalized by passing to a coset over $u(1)$, the coset not of the $\widehat{s\ell}(2)_k$ algebra as in the nonlogarithmic case but of the extended algebra $W(k)$ of the logarithmic model. Instead of working out the details of the resulting “logarithmic parafermion” model starting from representation theory, which seems to be quite a laborious task (cf. [27] in the nonlogarithmic case), we work at the level of characters, and this is how the \mathcal{A} and \mathcal{B} functions appear. The logarithmically extended parafermion model is, strictly speaking, presently nonexistent beyond as much as can be deduced from its proposed characters and the modular group representation generated from them, derived in what follows.

1.2. Technical issues. In contrast to the case with the standard string functions, our starting point is given by the characters not of (the integrable) $\widehat{s\ell}(2)_k$ -representations but of representations of the extended W -algebra $W(k)$ constructed in [2]. The integrable $\widehat{s\ell}(2)_k$ characters are quasiperiodic and holomorphic, but the $W(k)$ characters are neither. The integrable $\widehat{s\ell}(2)_k$ characters can therefore be decomposed with respect to a basis of level- k theta functions, yielding the string functions as the decomposition coefficients, but the $W(k)$ -characters require a larger basis for decomposition and hence yield more functions as the coefficients.

- First, the $W(k)$ characters are expressed in terms of theta-functions $\theta_{r,p}(q, z)$ and their derivatives $\theta'_{r,p}(q, z)$ and $\theta''_{r,p}(q, z)$; in the decomposition, this leads to the occurrence of $\theta_{n,k}(q, z)$, $\theta'_{n,k}(q, z)$, and $\theta''_{n,k}(q, z)$, the coefficients being $\mathcal{A}_{n,r}(q)$, $\mathcal{B}_{n,r}(q)$, and $\mathcal{C}_{n,r}(q)$. For the higher string functions, the analogue of the well-known periodicity $\mathcal{C}_{n+2k\ell,r}(q) = \mathcal{C}_{n,r}(q)$ takes a rather remarkable form: shifting $n \rightarrow n + 2k\ell$ gives rise to additional terms containing the triplet $\mathcal{W}(p)$ -algebra characters $\psi_r^\pm(q)$, with $p = k + 2$. For example,²

$$(1.3) \quad \mathcal{B}_{n+2k,r}(q) = \mathcal{B}_{n,r}(q) + \frac{\psi_r^-(q)}{\eta(q)} q^{-\frac{k}{4}(\frac{n}{k}+1)^2} - \frac{\psi_r^+(q)}{\eta(q)} q^{-\frac{k}{4}(\frac{n}{k}+2)^2}.$$

Generalizations of the “reflection” symmetry $\mathcal{C}_{-n,r}(q) = \mathcal{C}_{n,r}(q)$ also involve these characters, for example,

$$(1.4) \quad \mathcal{B}_{-n,r}(q) = -\mathcal{B}_{n,r}(q) - \frac{\psi_r^+(q)}{\eta(q)} q^{-\frac{n^2}{4k}}.$$

²The occurrence of $\mathcal{W}(p)$ characters may not be very surprising considering that the $\mathcal{B}_{n,r}$ “remember” their origin from the $W(k)$ algebra whose Hamiltonian reduction is just the $\mathcal{W}(p)$ algebra [2].

- Second, because the $W(k)$ characters are not holomorphic, they cannot decompose with respect to theta functions alone; in addition to $\theta_{n,k}(q, z)$ and their derivatives, the decomposition involves their meromorphic counterparts, the level- k Appell functions [1] (also see [36, 37])

$$\mathcal{K}_k(q^2, z, y) = \sum_{m \in \mathbb{Z}} \frac{q^{m^2 k} z^{mk}}{1 - zyq^{2m}}.$$

Under modular S -transformations, they behave as

$$(1.5) \quad \mathcal{K}_k\left(-\frac{1}{\tau}, \frac{\nu}{\tau}, \frac{\mu}{\tau}\right) = \tau e^{i\pi k \frac{\nu^2 - \mu^2}{\tau}} \mathcal{K}_k(\tau, \nu, \mu) + \tau \sum_{n=0}^{k-1} e^{i\pi \frac{k}{\tau} (\nu + \frac{n}{k} \tau)^2} \Phi(k\tau, k\mu - n\tau) \vartheta(k\tau, k\nu + n\tau),$$

which is the origin of the Φ function.³ In the decomposition of $W(k)$ characters, the coefficients at the Appell functions are just the $\mathcal{W}(p)$ characters $\psi_r^\pm(\tau)$.

To summarize, the $W(k)$ characters, as functions of z , decompose with respect to level- k theta functions and their first and second derivatives, and level- k Appell functions and their first derivatives. The decomposition coefficients, which are to be considered the “log-parafermionic” characters, are

$$(1.6) \quad (\psi_r^\pm(\tau), \mathcal{C}_{n,r}(\tau), \mathcal{B}_{n,r}(\tau), \mathcal{A}_{n,r}(\tau))$$

with $1 \leq r \leq p = k + 2$ and $0 \leq n \leq k$, “modulo” several relations at the range boundaries, such as $\mathcal{C}_{n,p}(\tau) = 0$, $\mathcal{B}_{0,r}(\tau) = -\frac{\psi_r^+(\tau)}{2\eta(\tau)}$, and $\mathcal{B}_{k,r}(\tau) = \frac{\psi_r^-(\tau)}{2\eta(\tau)} - \frac{\psi_r^+(\tau)}{\eta(\tau)} e^{-i\pi \frac{k}{2} \tau}$, together with $\mathcal{C}_{n+k,p-r}(\tau) = \mathcal{C}_{n,r}(\tau)$ (the ψ_r^\pm actually occur in the combinations $\psi_r^\pm(\tau) e^{-i\pi \frac{n^2}{2k} \tau} / \eta(\tau)$).

1.3. Results. The modular group representation generated from the set (1.6) follows from the modular transformations of the $W(k)$ -algebra characters in [2] and of the Appell functions in [1]. The simple modular transformation properties of $\mathcal{C}_{n,r}(\tau)$ and $\psi_r^\pm(\tau)$ characters are of course well known [24, 22], but S -transforms of $\mathcal{B}_{n,r}(\tau)$ and $\mathcal{A}_{n,r}(\tau)$ are new and turn out to *involve $\psi_r^\pm(\tau)$ times the Φ function*.

1.3.1. Notation. We fix an integer $k \geq 1$ and set

$$p = k + 2.$$

The reader is asked to excuse our mixed use of k and p , which sometimes both occur in

³That the level- k Appell functions, which were introduced and studied in [1] motivated by their occurrence in some characters of the affine Lie superalgebra $\widehat{sl}(2|1)$, make their appearance as “decomposition basis” elements in the $\widehat{sl}(2)/u(1)$ context may of course be attributed to the identification (in the supposedly rational case at least, see, e.g., [38])

$$\frac{\widehat{sl}(2)_k}{u(1)} = \frac{\widehat{sl}(2|1)_{k'}}{\widehat{gl}(2)_{k'}}, \quad (k+1)(k'+1) = 1.$$

the same formula; we frequently use $(-1)^k = (-1)^p$, $k+1 = p-1$, and other helpful identities. We also use the notation

$$\bar{a} = (a \bmod 2) \in \{0, 1\}$$

for any $a \in \mathbb{Z}$, and, more generally, $[a]_\ell = (a \bmod \ell) \in \{0, 1, \dots, \ell-1\}$.

We resort to the standard abuse by writing $f(\tau, \nu, \mu)$ for $f(e^{2i\pi\tau}, e^{2i\pi\nu}, e^{2i\pi\mu})$; it is tacitly assumed that $q = e^{2i\pi\tau}$ (with τ in the upper complex half-plane), $y = e^{2i\pi\mu}$, etc.

1.3.2. Background. We first quote the S -transform of the triplet $\mathcal{W}(p)$ algebra characters [22, 23]:

$$(1.7) \quad \psi_r^+(-\frac{1}{\tau}) = \sqrt{\frac{2}{p}} \frac{r}{2p} ((-1)^r \psi_p^+(\tau) + \psi_p^-(\tau)) \\ + \sqrt{\frac{2}{p}} \sum_{s=1}^{p-1} \frac{1}{i} S_{rs}(\tau) (\psi_s^+(\tau) + (-1)^r \psi_s^-(\tau)),$$

$$(1.8) \quad \psi_r^-(-\frac{1}{\tau}) = \sqrt{\frac{2}{p}} \frac{r}{2p} ((-1)^{p+r} \psi_p^+(\tau) + \psi_p^-(\tau)) \\ + \sqrt{\frac{2}{p}} \sum_{s=1}^{p-1} (-1)^s \frac{1}{i} S_{rs}(\tau) (\psi_s^+(\tau) + (-1)^{p+r} \psi_s^-(\tau)),$$

where

$$(1.9) \quad S_{rs}(\tau) = \frac{ir}{p} \cos \frac{\pi rs}{p} + \tau \frac{p-s}{p} \sin \frac{\pi rs}{p}.$$

A notable feature of logarithmic conformal field theory is the explicit occurrence of τ here. We next recall that the string functions $\mathcal{C}_{m,r}(\tau)$ with $\bar{m} = \overline{r+1}$ S -transform as [24]

$$(1.10) \quad \mathcal{C}_{m,r}(-\frac{1}{\tau}) = \frac{1}{\sqrt{pk}} \sum_{s=1}^{k+1} \sum_{\substack{n=0 \\ \bar{n}=s+1}}^{2k-1} e^{i\pi \frac{mn}{k}} \sin \frac{\pi rs}{p} \mathcal{C}_{n,s}(\tau).$$

The next theorem shows a nontrivial “merger” of the above formulas, additionally incorporating Φ , in the S -transformation of $\mathcal{B}_{m,r}(\tau)$.

1.3.3. Theorem. For $1 \leq r \leq p$ and $\bar{m} = \overline{r+1}$, let

$$(1.11) \quad \mathcal{B}_{m,r}(\tau) = \mathcal{B}_{m,r}(\tau) - i \frac{\psi_r^+(\tau) - \psi_r^-(\tau)}{2\sqrt{-2ik\tau}\eta(\tau)} \\ - \frac{\psi_r^+(\tau)}{\eta(\tau)} \Phi(2k\tau, m\tau) + \frac{\psi_r^-(\tau)}{\eta(\tau)} \Phi(2k\tau, (m-k)\tau).$$

Then

$$(1.12) \quad \mathcal{B}_{m+2k,r}(\tau) = \mathcal{B}_{m,r}(\tau), \\ \mathcal{B}_{-m,r}(\tau) + \mathcal{B}_{m,r}(\tau) = 0$$

and

$$\mathcal{B}_{m,r}(-\frac{1}{\tau}) = \frac{(-1)^r}{\sqrt{kp}} \frac{2r}{p} \sum_{\substack{n=1 \\ \bar{n}=k+1}}^{k-1} \sin \frac{\pi mn}{k} \mathcal{B}_{-n,p}(\tau) - \frac{4i}{\sqrt{kp}} \sum_{n=1}^{k-1} \sum_{\substack{s=1 \\ \bar{n}+s=1}}^{p-1} \sin \frac{\pi mn}{k} S_{rs}(\tau) \mathcal{B}_{-n,s}(\tau).$$

The Φ functions involved in the S -transformation are thus neatly incorporated in the definition of the “ Φ -modified” string functions (1.11), for which the properties such as (1.3) and (1.4) are “improved,” to become the respective relations in (1.12), and the S -transform formula takes the simplest form. We note that $B_{m,r}(-\frac{1}{\tau})$ (and hence $B_{m,r}(-\frac{1}{\tau})$) with $1 \leq m \leq k-1$ are expressed through $B_{n,s}(\tau)$ with $-k+1 \leq n \leq -1$; to reexpress the right-hand side in terms of positive-moded $B_{n,s}(\tau)$, Eqs. (1.3)–(1.4) must be used; evidently, expressing the right-hand side in terms of the $B_{m,s}(\tau)$ introduces the Φ functions. Iterating the S -transformation, in terms of either \mathcal{B} or B , inevitably leads to accumulating Φ ’s with different arguments, and it is clear that the modular group relations require that certain such combinations evaluate in terms of elementary functions (exponentials). Because Φ itself originates from modular transformation (1.5), it satisfies the necessary “consistency” conditions, as is detailed in [1]; specifically in the string-function context, the relevant identities are explicitly given in **B.3.3** in what follows.

The above S -transformation may be compared with (1.10), suggestively rewritten as

$$\mathcal{C}_{m,r}(-\frac{1}{\tau}) = \frac{2}{\sqrt{pk}} \sum_{n=0}^{k-1} \sum_{\substack{s=1 \\ n+s=1}}^{p-1} \cos \frac{\pi mn}{k} \sin \frac{\pi rs}{p} \mathcal{C}_{-n,s}(\tau).$$

Besides $\sin \frac{\pi mn}{k}$ in the theorem replacing $\cos \frac{\pi mn}{k}$ in the above formula (in accordance with the “odd” property of B in (1.12)), a notable difference is that τ explicitly occurs in $S_{rs}(\tau)$, a feature in common with the $(p, 1)$ logarithmic model; but the most essential increase in complexity in passing to the \mathcal{B} case is the incorporation of the Φ function in (1.11).

We also note that $B_{0,r}(q)$ and $B_{k,r}(q)$ defined as in (1.11) vanish, which means that $B_{k\ell,r}(q)$, $\ell \in \mathbb{Z}$, are in $\mathbb{C}[\psi_r^\pm(\tau) e^{-i\pi \frac{n^2}{2k} \tau} / \eta(\tau)]$ ($n \in \mathbb{Z}$). The S -transform formula in the theorem is therefore consistent but not informative for $m = 0$, k .

The proof of the S -transform formula in **1.3.3** is the content of **3.2**; simple relations (1.12) are shown in Appendix **D**.

For $\mathcal{A}_{n,r}$, the counterparts of relations (1.3) and (1.4) are

$$(1.13) \quad \mathcal{A}_{n+2k,r}(q) = \mathcal{A}_{n,r}(q) - \left(1 + \frac{n}{k}\right) \frac{\psi_r^-(q)}{\eta(q)} q^{-\frac{(n+k)^2}{4k}} + \left(2 + \frac{n}{k}\right) \frac{\psi_r^+(q)}{\eta(q)} q^{-\frac{(n+2k)^2}{4k}}$$

and

$$(1.14) \quad \mathcal{A}_{-n,r}(q) = \mathcal{A}_{n,r}(q) - \frac{n}{k} \frac{\psi_r^+(q)}{\eta(q)} q^{-\frac{n^2}{4k}}.$$

As with the \mathcal{B} , these properties are “improved” for Φ -modified string functions. We set

$$(1.15) \quad \Phi'(\tau, \mu) = \frac{1}{2i\pi} \frac{\partial}{\partial \mu} \Phi(\tau, \mu).$$

1.3.4. Theorem. For $1 \leq r \leq p$ and $\overline{m} = \overline{r+1}$, let

$$(1.16) \quad A_{m,r}(\tau) = \mathcal{A}_{m,r}(\tau) - \frac{2\psi_r^+(\tau)}{\eta(\tau)} \Phi'(2k\tau, m\tau) + \frac{2\psi_r^-(\tau)}{\eta(\tau)} \Phi'(2k\tau, (m-k)\tau).$$

Then

$$(1.17) \quad \begin{aligned} A_{m+2k,r}(\tau) &= A_{m,r}(\tau), \\ A_{-m,r}(\tau) - A_{m,r}(\tau) &= 0, \end{aligned}$$

and

$$\begin{aligned} A_{m,r}\left(-\frac{1}{\tau}\right) &= \\ &= \frac{2\tau}{\sqrt{pk}} \left(\left(1 + (-1)^{k+1}\right) \frac{(-1)^r ir}{4p} \mathcal{A}_{0,p}(\tau) + \sum_{\substack{s=1 \\ \bar{s}=1}}^{p-1} S_{rs}(\tau) \mathcal{A}_{0,s}(\tau) - (-1)^r \sum_{\substack{s=1 \\ \bar{s}=k+1}}^{p-1} S_{rs}(\tau) \mathcal{A}_{-k,s}(\tau) \right) \\ &\quad + \frac{4\tau}{\sqrt{pk}} \sum_{\substack{n=1 \\ \bar{n}=k+1}}^{k-1} \cos \frac{\pi mn}{k} \frac{(-1)^r ir}{2p} \mathcal{A}_{-n,p}(\tau) + \frac{4\tau}{\sqrt{pk}} \sum_{\substack{n=1 \\ \bar{n}=s=1}}^{k-1} \sum_{\substack{s=1 \\ \bar{n}+\bar{s}=1}}^{p-1} \cos \frac{\pi mn}{k} S_{rs}(\tau) \mathcal{A}_{-n,s}(\tau) \\ &\quad + \frac{1}{\sqrt{pk}} \sum_{\substack{n=0 \\ \bar{n}=s=1}}^{k-1} \sum_{\substack{s=1 \\ \bar{n}+\bar{s}=1}}^{p-1} \cos \frac{\pi mn}{k} U_{rs}(\tau) \mathcal{C}_{n,s}(\tau), \end{aligned}$$

where

$$U_{rs}(\tau) = \frac{ir(p-2s)\tau}{2p^2} \cos \frac{\pi rs}{p} + \left(\frac{s(s-p)\tau^2}{2p^2} - \frac{\tau}{i\pi pk} + \frac{r^2}{2p^2} \right) \sin \frac{\pi rs}{p}.$$

This formula looks more complicated than its “lower” analogue in **1.3.3** for three reasons: $A_{k\ell,r}$, $\ell \in \mathbb{Z}$, do not vanish and hence contribute to the transformation; also, the “ $\cos \frac{\pi mn}{k}$ ” representation of $SL(2, \mathbb{Z})$ is somewhat bulkier than the “ $\sin \frac{\pi mn}{k}$ ” representation (when $\mathcal{A}_{0,*}$ are not related to $\mathcal{A}_{\pm k,*}$); finally, there is an “admixture” of the \mathcal{C} string functions.

The proof of the S -transform formula is the content of **3.2**; simple relations (1.17) are shown in Appendix **D**.

1.3.5. We note that the T -transformation $\tau \mapsto \tau + 1$ amounts to multiplying $\mathcal{A}_{m,r}(\tau)$, $\mathcal{B}_{m,r}(\tau)$, and $\mathcal{C}_{m,r}(\tau)$ by $e^{i\pi(\frac{r^2}{2p} - \frac{r^2}{2k} - \frac{1}{6})}$ and $\frac{\psi_r^\pm(\tau)}{\eta(\tau)}$ by $e^{i\pi(\frac{r^2}{2p} - \frac{1}{6})}$.

Plan of the paper. We extract the higher string functions from decomposing the characters of the triplet W -algebra $W(k)$ of logarithmically extended $\widehat{sl}(2)_k$ -models in Sec. **2**. Modular S -transformations of the higher string functions are derived in Sec. **3**. Theta-function conventions are fixed in Appendix **A**. The necessary properties of the Appell functions are recalled in Appendix **B**. The $W(k)$ -algebra characters are listed and their modular properties are recalled in Appendix **C**. Some simple properties of the higher string functions are derived in Appendix **D**.

The calculations leading to the results stated above are straightforward but quite bulky. Besides, the Appell functions K and the related Φ function are integrated into the derivation, and their properties have a considerable impact on the “calculation flow,” with the “sign” of the effect dependent on whether these properties are used timely or untimely.

Essential simplification (although possibly still far from the ideal) is achieved by consolidating the relevant K/Φ properties in **B.3.1**.

2. CHARACTER DECOMPOSITIONS

In this section, we establish the decomposition, or “branching,” of the $W(k)$ -algebra characters in [2]. The method is very direct and is based on the identity (see [39, 24] and the references therein)

$$(2.1) \quad \frac{1}{q^{\frac{1}{8}} \vartheta_{1,1}(q, z)} = \frac{1}{\eta(q)^3} \sum_{m \in \mathbb{Z}} \sum_{j \geq 1} (-1)^{j+1} q^{\frac{1}{2}j(j-1) - jm} z^{-m}.$$

2.1. $\widehat{sl}(2)$ integrable representation characters. We first recall the classic result [24, 25] that the integrable $\widehat{sl}(2)_k$ characters

$$(2.2) \quad \chi_r(q, z) = \frac{\theta_{r,p}(q, z) - \theta_{-r,p}(q, z)}{\Omega(q, z)}, \quad r = 1, \dots, k+1,$$

decompose with respect to level- k theta-functions as

$$(2.3) \quad \chi_r(q, z) = -\frac{1}{\eta(q)} \sum_{\substack{n=0 \\ \bar{n}=r+1}}^{2k-1} \mathcal{C}_{n,r}(q) \theta_{n,k}(q, z).$$

Theta-function conventions and the definition of $\Omega(q, z)$ are given in Appendix A.

We next decompose the other $W(k)$ -characters similarly to (2.3).

2.2. Decomposition of the $W(k)$ -algebra characters.

2.2.1. The characters. In the logarithmic $\widehat{sl}(2)_k$ model for each $k = 0, 1, 2, \dots$, characters of the extended algebra $W(k)$ were calculated in [2]. The characters $\chi_r^\pm(q, z)$ are given by

$$(2.4) \quad \chi_r^+(q, z) = \frac{1}{\Omega(q, z)} \left(\frac{r^2}{4p^2} (\theta_{-r,p}(q, z) - \theta_{r,p}(q, z)) \right. \\ \left. + \frac{r}{p^2} (\theta'_{-r,p}(q, z) + \theta'_{r,p}(q, z)) + \frac{1}{p^2} (\theta''_{-r,p}(q, z) - \theta''_{r,p}(q, z)) \right),$$

$$(2.5) \quad \chi_r^-(q, z) = \frac{1}{\Omega(q, z)} \left(\left(\frac{r^2}{4p^2} - \frac{1}{4} \right) (\theta_{p-r,p}(q, z) - \theta_{p+r,p}(q, z)) \right. \\ \left. + \frac{r}{p^2} (\theta'_{p-r,p}(q, z) + \theta'_{p+r,p}(q, z)) + \frac{1}{p^2} (\theta''_{p-r,p}(q, z) - \theta''_{p+r,p}(q, z)) \right)$$

for $1 \leq r \leq p-1$ and

$$\chi_p^+(q, z) = \frac{2\theta'_{p,p}(q, z)}{p\Omega(q, z)}, \quad \chi_p^-(q, z) = \frac{2\theta'_{0,p}(q, z)}{p\Omega(q, z)}.$$

Under the spectral flow (see **C.1**), the $\chi_r^\pm(q, z)$ further generate $\omega_r^\pm(q, z)$ given by [2]

$$\omega_r^+(q, z) = \frac{1}{\Omega(q, z)} \left(\frac{r}{2p} (\theta_{r,p}(q, z) + \theta_{-r,p}(q, z)) - \frac{1}{p} (\theta'_{r,p}(q, z) - \theta'_{-r,p}(q, z)) \right),$$

$$\omega_r^-(q, z) = \frac{1}{\Omega(q, z)} \left(\frac{r}{2p} (\theta_{p-r,p}(q, z) + \theta_{r-p,p}(q, z)) - \frac{1}{p} (\theta'_{r-p,p}(q, z) - \theta'_{p-r,p}(q, z)) \right)$$

for $1 \leq r \leq p-1$, and

$$\omega_p^+(q, z) = \frac{\theta_{p,p}(q, z)}{\Omega(q, z)}, \quad \omega_p^-(q, z) = \frac{\theta_{0,p}(q, z)}{\Omega(q, z)}.$$

The characters decompose with respect to level- k theta and Appell functions and their derivatives. We set

$$\mathcal{K}'_{\alpha,k}(q, x, y) = (x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}) \mathcal{K}_{\alpha,k}(q, x, y),$$

where the functions $\mathcal{K}_{\alpha,k}(q, x, y)$ defined in (B.2).

2.2.2. Lemma. *As functions of z , the $W(k)$ characters $\chi_r^\pm(q, z)$ decompose with respect to level- k theta and Appell functions and their derivatives as*

$$\begin{aligned} \chi_r^+(q, z) = & \frac{1}{\eta(q)} \sum_{\substack{n=0 \\ \bar{n}=r+1}}^{2k-1} \left(\mathcal{A}_{n,r}(q) \theta_{n,k}(q, z) + \frac{2}{k} \mathcal{B}_{n,r}(q) \theta'_{n,k}(q, z) + \frac{1}{k^2} \mathcal{C}_{n,r}(q) \theta''_{n,k}(q, z) \right) \\ & - \frac{2\psi_r^+(q)}{\eta(q)^2} q^{-k} \left(\frac{1}{k} \mathcal{K}'_{r+1,k}(q, z, q^{-2}) - \mathcal{K}_{r+1,k}(q, z, q^{-2}) \right) \\ & + \frac{\psi_r^-(q)}{\eta(q)^2} q^{-\frac{k}{4}} \left(\frac{2}{k} \mathcal{K}'_{r+1,k}(q, z, q^{-1}) - \mathcal{K}_{r+1,k}(q, z, q^{-1}) \right) \end{aligned}$$

and

$$\begin{aligned} \chi_r^-(q, z) = & -\frac{1}{\eta(q)} \sum_{\substack{n=0 \\ \bar{n}=r+1}}^{2k-1} \left(\left(\mathcal{A}_{n,r}(q) - \frac{1}{4} \mathcal{C}_{n,r}(q) \right) \theta_{n+k,k}(q, z) \right. \\ & \left. + \frac{2}{k} \mathcal{B}_{n,r}(q) \theta'_{n+k,k}(q, z) + \frac{1}{k^2} \mathcal{C}_{n,r}(q) \theta''_{n+k,k}(q, z) \right) \\ & + \frac{\psi_r^+(q)}{\eta(q)^2} q^{-\frac{k}{4}} \left(\frac{2}{k} \mathcal{K}'_{k+r+1,k}(q, z, q^{-1}) - \mathcal{K}_{k+r+1,k}(q, z, q^{-1}) \right) \\ & - \frac{\psi_r^-(q)}{\eta(q)^2} \frac{2}{k} \mathcal{K}'_{k+r+1,k}(q, z, 1). \end{aligned}$$

A simple corollary follows if we use (C.3) to evaluate $\omega_r^+(q, z) = -\chi_{r;1}^-(q, z) - \chi_r^+(q, z)$ and $\omega_r^-(q, z) = -\chi_{r;1}^+(q, z) - \chi_r^-(q, z) - \frac{1}{2} \chi_{p-r}(q, z)$ with the above decompositions of χ_r^\pm and χ_r .

2.2.3. Corollary. *There are the decompositions*

$$\begin{aligned} \omega_r^+(q, z) = & \frac{1}{\eta(q)} \sum_{\substack{n=0 \\ \bar{n}=r+1}}^{2k-1} \left(\mathcal{B}_{n,r}(q) \theta_{n,k}(q, z) + \frac{1}{k} \mathcal{C}_{n,r}(q) \theta'_{n,k}(q, z) \right) \\ & - \frac{\psi_r^+(q)}{\eta(q)^2} q^{-k} \mathcal{K}_{r+1,k}(q, z, q^{-2}) + \frac{\psi_r^-(q)}{\eta(q)^2} q^{-\frac{k}{4}} \mathcal{K}_{r+1,k}(q, z, q^{-1}) \end{aligned}$$

and

$$\begin{aligned} \omega_r^-(q, z) = & -\frac{1}{\eta(q)} \sum_{\substack{n=0 \\ \bar{n}=r+1}}^{2k-1} \left(\mathcal{B}_{n,r}(q) \theta_{n+k,k}(q, z) + \frac{1}{k} \mathcal{C}_{n,r}(q) \theta'_{n+k,k}(q, z) \right) \\ & + \frac{\psi_r^+(q)}{\eta(q)^2} q^{-\frac{k}{4}} \mathcal{K}_{k+r+1,k}(q, z, q^{-1}) - \frac{\psi_r^-(q)}{\eta(q)^2} \mathcal{K}_{k+r+1,k}(q, z, 1). \end{aligned}$$

2.3. Proof of 2.2.2. We derive the decomposition formula for $\chi_r^-(q, z)$ in 2.3.1 and the formula for $\chi_r^+(q, z)$ in 2.3.2.

2.3.1. $\chi_r^-(q, z)$. We write the χ_r^- character in (2.5) as

$$\chi_r^-(q, z) = \frac{1}{q^{\frac{r}{8}} \vartheta_{1,1}(q, z)} \sum_{a \in \mathbb{Z} + \frac{1}{2}} \left(a^2 - \frac{1}{4} \right) q^{p(\frac{r}{2p} + a)^2} \left(z^{-\frac{r+1}{2} - ap} - z^{\frac{r-1}{2} + ap} \right).$$

Using identity (2.1), we calculate

$$\begin{aligned} \chi_r^-(q, z) = & \frac{q^{\frac{r^2}{4p}}}{\eta(q)^3} \sum_{a \in \mathbb{Z} + \frac{1}{2}} \sum_{m \in \mathbb{Z}} \sum_{j \geq 1} (-1)^{j+1} \left(a^2 - \frac{1}{4} \right) q^{\frac{1}{2}j(j-1) - jm + pa^2 + ra} z^{-\frac{r}{2} - ap - m - \frac{1}{2}} - (r \mapsto -r). \end{aligned}$$

We now shift the summation variable as $a \mapsto a - \frac{1}{2}$ and then pass from summation over m to summation over $n = 2m + r + 2ap - 1$, which (with integer-valued a) ranges over $2\mathbb{Z} + r + 1$. Shifting $j \mapsto j + 1$ then yields

$$\begin{aligned} \chi_r^-(q, z) = & \frac{q^{\frac{r^2}{4p}}}{\eta(q)^3} \sum_{n \in 2\mathbb{Z} + r + 1} q^{\frac{k}{4} - \frac{n}{2}} z^{\frac{k}{2} - \frac{n}{2}} \sum_{a \in \mathbb{Z}} \sum_{j \geq 0} (-1)^j q^{\frac{1}{2}j(j-n)} (a^2 - a) q^{j(ap + \frac{r}{2}) + pa^2 + ra} - (r \mapsto -r). \end{aligned}$$

Next, the elementary identity

$$(2.6) \quad \sum_{j \in \mathbb{Z}} (-1)^j q^{\frac{1}{2}j(j-1) + jn} = 0, \quad n \in \mathbb{Z},$$

and the antisymmetry of the entire expression for χ_r^- under $r \mapsto -r$ allow us to conclude that

$$\chi_r^-(q, z) = -\frac{q^{\frac{r^2}{4p}}}{\eta(q)^3} \sum_{n \in 2\mathbb{Z} + r + 1} q^{\frac{k}{4} + \frac{n}{2}} z^{\frac{k}{2} + \frac{n}{2}} (A_{n,r}(q) + B_{n,r}(q))$$

with $A_{n,r}(q)$ and $B_{n,r}(q)$ defined in (D.3) and (D.2). The formulas in **D.1** for $A_{n+2k\ell,r}$ and $B_{n+2k\ell,r}$ then yield

$$\begin{aligned} (2.7) \quad \chi_r^-(q, z) = & -\frac{1}{\eta(q)} \sum_{\substack{n=0 \\ \bar{n}=r+1}}^{2k-1} \left((\mathcal{A}_{n,r}(q) - \frac{1}{4} \mathcal{C}_{n,r}(q)) \theta_{n+k,k}(q, z) \right. \\ & \left. + \frac{2}{k} \mathcal{B}_{n,r}(q) \theta'_{n+k,k}(q, z) + \frac{1}{k^2} \mathcal{C}_{n,r}(q) \theta''_{n+k,k}(q, z) \right) \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{\eta(q)^2} \sum_{\substack{n=0 \\ \bar{n}=r+1}}^{2k-1} \left(\sum_{j \geq 1} \sum_{\ell \geq j} - \sum_{j \leq 0} \sum_{\ell \leq j-1} \right) q^{\frac{k}{4} + \frac{n}{2} + k\ell^2 + k\ell + (n)\ell} z^{\frac{k}{2} + \frac{n}{2} + k\ell} \times \\
& \times \left((2\ell - 2j + 1) q^{-kj^2 - j(n)} \psi_r^+(q) - (2\ell - 2j + 2) q^{-\frac{k}{4}(2j-1)^2 - (j-\frac{1}{2})(n)} \psi_r^-(q) \right),
\end{aligned}$$

where

$$\begin{aligned}
(2.8) \quad \mathcal{C}_{n,r}(q) &= \frac{q^{\frac{r^2}{4p} - \frac{n^2}{4k}}}{\eta(q)^2} C_{n,r}(q), \\
\mathcal{B}_{n,r}(q) &= \frac{q^{\frac{r^2}{4p} - \frac{n^2}{4k}}}{\eta(q)^2} \left(B_{n,r}(q) - \frac{n}{2k} C_{n,r}(q) \right), \\
\mathcal{A}_{n,r}(q) &= \frac{q^{\frac{r^2}{4p} - \frac{n^2}{4k}}}{\eta(q)^2} \left(A_{n,r}(q) - \frac{n}{k} B_{n,r}(q) + \frac{n^2}{4k^2} C_{n,r}(q) \right),
\end{aligned}$$

In the “ ψ -part” of (2.7), we make the shift $\ell \mapsto \ell + j$, which produces the sums

$$\sum_{j \geq 1} \sum_{\ell \geq 0} - \sum_{j \leq 0} \sum_{\ell \leq -1} = \sum_{\ell \geq 0} \sum_{j \geq 1} - \sum_{\ell \leq -1} \sum_{j \leq 0},$$

and evaluate the resulting j -sums. Examination shows that under the condition

$$(2.9) \quad |q| < |z| < 1,$$

all of the j -sums are of the form $\sum_j \xi^j$ with $|\xi| < 1$, summed over positive (nonnegative) j . For each $\ell \leq -1$, for instance, the coefficient at $\psi_r^-(q)$ involves the sums

$$\sum_{j \leq 0} (q^{2k(\ell+1)} z^k)^j = \sum_{j \geq 0} (q^{2k(-\ell-1)} z^{-k})^j,$$

where $|q^{2k(-\ell-1)} z^{-k}| < 1$ for any $\ell \leq -2$. This estimate does not hold in the sole case $\ell = -1$, but the divergent sum $\sum_{j \geq 0} z^{-kj}$ does not actually occur because of the factor $(2\ell + 2)$ in front of $\psi_r^-(q)$ (after the shift $\ell \mapsto \ell + j$ in the “ ψ -part” in (2.7)). The result is

$$\begin{aligned}
\chi_r^-(q, z) &= -\frac{1}{\eta(q)} \sum_{\substack{n=0 \\ \bar{n}=r+1}}^{2k-1} \left((\mathcal{A}_{n,r}(q) - \frac{1}{4} \mathcal{C}_{n,r}(q)) \theta_{n+k,k}(q, z) \right. \\
&\quad \left. + \frac{2}{k} \mathcal{B}_{n,r}(q) \theta'_{n+k,k}(q, z) + \frac{1}{k^2} \mathcal{C}_{n,r}(q) \theta''_{n+k,k}(q, z) \right) \\
&\quad + \frac{1}{\eta(q)^2} \sum_{\ell \in \mathbb{Z}} q^{\frac{k}{4} + \frac{1}{2}(r+1) + k\ell^2 + k\ell + (r+1)\ell} q^{2(\ell+1)k} z^{\frac{k}{2} + \frac{1}{2}(r+1) + k\ell + k} \times \\
&\quad \times \left(\frac{(2\ell+1)q^{-k}}{1 - zq^{2\ell+1}} \psi_r^+(q) - \frac{(2\ell+2)q^{-\frac{k}{4} + \frac{1}{2}(r+1)}}{1 - zq^{2(\ell+1)}} \psi_r^-(q) \right).
\end{aligned}$$

After simple rearrangements, we obtain the formula in the theorem.

2.3.2. $\chi_r^+(q, z)$. We write the χ_r^+ character as

$$\chi_r^+(q, z) = \frac{1}{q^{\frac{1}{8}} \vartheta_{1,1}(q, z)} \sum_{a \in \mathbb{Z}} a^2 q^{p(\frac{r}{2p} + a)^2} \left(z^{-\frac{r+1}{2} - ap} - z^{\frac{r-1}{2} + ap} \right).$$

Using (2.1) again, we find

$$\begin{aligned}\chi_r^+(q, z) &= \frac{1}{\eta(q)^3} \sum_{m \in \mathbb{Z}} z^{-m-\frac{1}{2}} \sum_{j \geq 1} (-1)^{j+1} q^{\frac{1}{2}j(j-1)-jm} \sum_{a \in \mathbb{Z}} \sum_{\sigma = \pm 1} \sigma a^2 q^{p(\frac{r}{2p}+a)^2} z^{-\sigma \frac{r}{2}-\sigma ap} \\ &= \frac{q^{\frac{r^2}{4p}}}{\eta(q)^3} \sum_{n \in 2\mathbb{Z}+r+1} z^{-\frac{n}{2}} A_{n,r}(q),\end{aligned}$$

with $A_{n,r}(q)$ defined in (D.3).

Next, identity (2.6) shows that

$$\begin{aligned}A_{n,r} - A_{-n,r} &= \sum_{a \in \mathbb{Z}} a^2 \left(\sum_{j \geq 1} + \sum_{j \leq -1} \right) (-1)^{j+1} q^{\frac{1}{2}j(j-n)+ra+pa^2} \left(q^{\frac{1}{2}j(2ap+r)} - q^{-\frac{1}{2}j(2ap+r)} \right) = 0,\end{aligned}$$

and therefore

$$\sum_{n \in 2\mathbb{Z}+r+1} z^{-\frac{n}{2}} A_{n,r}(q) = \sum_{n \in 2\mathbb{Z}+r+1} z^{\frac{n}{2}} A_{n,r}(q) = \sum_{\substack{n=0 \\ \bar{n}=r+1}}^{2k-1} \sum_{\ell \in \mathbb{Z}} z^{k\ell+\frac{n}{2}} A_{2k\ell+n,r}.$$

Finally, the formula for $A_{n+2k\ell,r}(q)$ in **D.1** allows obtaining

$$\begin{aligned}\chi_r^+(q, z) &= \frac{1}{\eta(q)} \sum_{\substack{n=0 \\ \bar{n}=r+1}}^{2k-1} \left(\mathcal{A}_{n,r}(q) \theta_{n,k}(q, z) + \frac{2}{k} \mathcal{B}_{n,r}(q) \theta'_{n,k}(q, z) + \frac{1}{k^2} \mathcal{C}_{n,r}(q) \theta''_{n,k}(q, z) \right) \\ &\quad - \frac{z^{\frac{r-1}{2}}}{\eta(q)^2} \sum_{\ell \in \mathbb{Z}} z^{k(\ell+1)} q^{k(\ell+1)^2+\ell(\bar{r}+1)} \left(\frac{2\ell q^{-k}}{1-zq^{2\ell}} \psi_r^+(q) - \frac{(2\ell+1)q^{-\frac{k}{4}+\frac{1}{2}(\bar{r}+1)}}{1-zq^{2\ell+1}} \psi_r^-(q) \right)\end{aligned}$$

(again, with $\mathcal{C}_{n,r}(q)$, $\mathcal{B}_{n,r}(q)$, and $\mathcal{A}_{n,r}(q)$ expressed as in (2.8)), which readily yields the formula in **2.2.2**.

2.3.3. Remark. It is easy to see that there is an equivalent representation for $\chi_r^-(q, z)$ and $\omega_r^-(q, z)$, with the Appell-function characteristics “normalized” to $\{0, 1\}$:

$$\begin{aligned}\chi_r^-(q, z) &= -\frac{1}{\eta(q)} \sum_{\substack{n=0 \\ \bar{n}=r+k+1}}^{2k-1} \left(\left(\mathcal{A}_{n-k,r}(q) - \frac{1}{4} \mathcal{C}_{n-k,r}(q) \right) \theta_{n,k}(q, z) \right. \\ &\quad \left. + \frac{2}{k} \mathcal{B}_{n-k,r}(q) \theta'_{n,k}(q, z) + \frac{1}{k^2} \mathcal{C}_{n-k,r}(q) \theta''_{n,k}(q, z) \right) \\ &\quad + \frac{\psi_r^+(q)}{\eta(q)^2} q^{-\frac{k}{4}} \left(\frac{2}{k} \mathcal{K}'_{k+r+1,k}(q, z, q^{-1}) - \mathcal{K}_{k+r+1,k}(q, z, q^{-1}) \right) - \frac{\psi_r^-(q)}{\eta(q)^2} \frac{2}{k} \mathcal{K}'_{k+r+1,k}(q, z, 1)\end{aligned}$$

and

$$\begin{aligned}\omega_r^-(q, z) &= -\frac{1}{\eta(q)} \sum_{\substack{n=0 \\ \bar{n}=k+r+1}}^{2k-1} \left(\mathcal{B}_{n-k,r}(q) \theta_{n,k}(q, z) + \frac{1}{k} \mathcal{C}_{n-k,r}(q) \theta'_{n,k}(q, z) \right) \\ &\quad + \frac{\psi_r^+(q)}{\eta(q)^2} q^{-\frac{k}{4}} \mathcal{K}_{k+r+1,k}(q, z, q^{-1}) - \frac{\psi_r^-(q)}{\eta(q)^2} \mathcal{K}_{k+r+1,k}(q, z, 1).\end{aligned}$$

3. MODULAR TRANSFORMATIONS

In this section, we use the decompositions in **2.2.2** and **2.2.3** to derive modular transformation properties of the functions (1.6) occurring there as coefficients, among which we are interested in the string functions $\mathcal{B}_{n,r}(\tau)$ and $\mathcal{A}_{n,r}(\tau)$; that is, we prove the S -transformation formulas in **1.3.3** and **1.3.4**.

3.1. $\mathcal{C}_{n,r}(-\frac{1}{\tau})$. For uniformity, we first rederive the well-known S -transformation of the string functions $\mathcal{C}_{n,r}(\tau)$. From (2.3), (A.6), and (A.13),

$$\chi_r(-\frac{1}{\tau}, \frac{v}{\tau}) = -\frac{1}{\sqrt{2k}} \frac{e^{i\pi k \frac{v^2}{2\tau}}}{\eta(\tau)} \sum_{\substack{m=0 \\ \bar{m}=r+1}}^{2k-1} \mathcal{C}_{m,r}(-\frac{1}{\tau}) \sum_{n=0}^{2k-1} e^{-i\pi \frac{mn}{k}} \theta_{n,k}(\tau, v),$$

but in view of (C.8) this is simultaneously equal to

$$= -\sqrt{\frac{2}{p}} \frac{e^{i\pi k \frac{v^2}{2\tau}}}{\eta(\tau)} \sum_{s=1}^{p-1} \sin \frac{\pi rs}{p} \sum_{\substack{n=0 \\ \bar{n}=s+1}}^{2k-1} \mathcal{C}_{n,s}(\tau) \theta_{n,k}(\tau, v).$$

Comparing the two expressions immediately yields (1.10).

3.2. $\mathcal{B}_{n,r}(-\frac{1}{\tau})$. Following the same simple strategy to find $\mathcal{B}_{n,r}(-\frac{1}{\tau})$ is somewhat more involved. It is technically convenient to introduce the linear combinations

$$\begin{aligned} \Omega_r^a(\tau, v) &= \omega_r^+(\tau, v) + (-1)^a \omega_r^-(\tau, v), \\ \mathbb{K}_\alpha^a(\tau, v, \mu) &= K_{\alpha,k}(\tau, v, \mu) + (-1)^a e^{-i\pi k \frac{\tau}{2} - i\pi k \mu} K_{\alpha+k,k}(\tau, v, \mu + \tau). \end{aligned}$$

3.2.1. From **2.2.3**, (A.6), (A.7), and **B.3.1**, we calculate

$$\begin{aligned} \Omega_r^a(-\frac{1}{\tau}, \frac{v}{\tau}) &= \frac{2e^{i\pi k \frac{v^2}{2\tau}}}{\sqrt{2k}\eta(\tau)} \sum_{\substack{m=0 \\ \bar{m}=r+1}}^{2k-1} \sum_{\substack{n=0 \\ \bar{n}=a+1}}^{2k-1} e^{-i\pi \frac{mn}{k}} \left(\mathcal{B}_{m,r}(-\frac{1}{\tau}) \theta_{n,k}(\tau, v) \right. \\ &\quad \left. + \frac{1}{k} \mathcal{C}_{m,r}(-\frac{1}{\tau}) (\tau \theta'_{n,k}(\tau, v) + \frac{kv}{2} \theta_{n,k}(\tau, v)) \right) \\ &\quad + \frac{e^{i\pi k \frac{v^2}{2\tau}} \psi_r^+(-\frac{1}{\tau})}{i\eta(\tau)^2} \left[K_{\frac{a+1}{2},k}(\tau, v, 0) + (-1)^{r+1} e^{-i\pi k \frac{\tau}{2}} K_{\frac{a+1}{2},k}(\tau, v, -\tau) \right. \\ &\quad \left. + e^{2i\pi \frac{k}{\tau}} \sum_{\beta \in \{0,1\}} \sum_{\substack{n=0 \\ \bar{n}=a+1}}^{2k-1} (-1)^{(r+1)\beta} \Phi(2k\tau, 2k - n\tau - \beta k\tau) \theta_{n,k}(\tau, v) \right] \\ &\quad - \frac{e^{i\pi k \frac{v^2}{2\tau}} \psi_r^-(-\frac{1}{\tau})}{i\eta(\tau)^2} \left[(-1)^{a+1} K_{\frac{a+1}{2},k}(\tau, v, 0) + (-1)^{r+a+k} e^{-i\pi k \frac{\tau}{2}} K_{\frac{a+1}{2},k}(\tau, v, -\tau) \right. \\ &\quad \left. + e^{i\pi \frac{k}{2\tau}} \sum_{\beta \in \{0,1\}} \sum_{\substack{n=0 \\ \bar{n}=a+1}}^{2k-1} (-1)^{(r+1)\beta} \Phi(2k\tau, k - n\tau - \beta k\tau) \theta_{n,k}(\tau, v) \right]. \end{aligned}$$

3.2.2. On the other hand, it follows from **C.2.2** that

$$\begin{aligned} \Omega_r^a\left(-\frac{1}{\tau}, \frac{\nu}{\tau}\right) &= \sqrt{\frac{2}{p}} e^{i\pi k \frac{\nu^2}{2\tau}} \left(\frac{ir}{2p} \left[(-1)^r (1 + (-1)^{a+p}) \omega_p^+(\tau, \nu) + (1 + (-1)^a) \omega_p^-(\tau, \nu) \right] \right. \\ &\quad \left. + 2 \sum_{\substack{s=1 \\ \bar{s}=a}}^{p-1} S_{rs}(\tau) \omega_s^+(\tau, \nu) + 2(-1)^r \sum_{\substack{s=1 \\ \bar{s}=a+k}}^{p-1} S_{rs}(\tau) \omega_s^-(\tau, \nu) \right) \\ &\quad - \frac{2\nu}{\sqrt{2p}} e^{i\pi k \frac{\nu^2}{2\tau}} \sum_{\substack{s=1 \\ \bar{s}=a}}^{p-1} \sin \frac{\pi rs}{p} \chi_s(\tau, \nu) \end{aligned}$$

(see (1.9) for $S_{rs}(\tau)$)). We next use the decompositions of ω_s^\pm (and $\chi_s(\tau, \nu)$) again. More precisely, we express ω_s^+ from **2.2.3** and ω_s^- from **2.3.3**, which gives

$$\begin{aligned} \Omega_r^a\left(-\frac{1}{\tau}, \frac{\nu}{\tau}\right) &= \sqrt{\frac{2}{p}} \frac{e^{i\pi k \frac{\nu^2}{2\tau}}}{\eta(\tau)} \frac{ir}{2p} \times \\ &\times \left[(-1)^r (1 + (-1)^{a+k}) \sum_{\substack{n=0 \\ \bar{n}=a+1}}^{2k-1} \mathcal{B}_{n,p}(\tau) \theta_{n,k}(\tau, \nu) - (1 + (-1)^a) \sum_{\substack{n=0 \\ \bar{n}=a+1}}^{2k-1} \mathcal{B}_{n-k,p}(\tau) \theta_{n,k}(\tau, \nu) \right. \\ &\quad - (-1)^r (1 + (-1)^{a+k}) \left(\frac{\psi_p^+(\tau) e^{-2i\pi k \tau}}{\eta(\tau)} K_{\overline{a+1},k}(\tau, \nu, -2\tau) - \frac{\psi_p^-(\tau) e^{-i\pi \frac{k}{2} \tau}}{\eta(\tau)} K_{\overline{a+1},k}(\tau, \nu, -\tau) \right) \\ &\quad \left. + (1 + (-1)^a) \left(\frac{\psi_p^+(\tau) e^{-i\pi \frac{k}{2} \tau}}{\eta(\tau)} K_{\overline{a+1},k}(\tau, \nu, -\tau) - \frac{\psi_p^-(\tau)}{\eta(\tau)} K_{\overline{a+1},k}(\tau, \nu, 0) \right) \right] \\ &+ 2 \sqrt{\frac{2}{p}} \frac{e^{i\pi k \frac{\nu^2}{2\tau}}}{\eta(\tau)} \sum_{\substack{s=1 \\ \bar{s}=a}}^{p-1} S_{rs}(\tau) \left[\sum_{\substack{n=0 \\ \bar{n}=a+1}}^{2k-1} \left(\mathcal{B}_{n,s}(\tau) \theta_{n,k}(\tau, \nu) + \frac{1}{k} \mathcal{C}_{n,s}(\tau) \theta'_{n,k}(\tau, \nu) \right) \right. \\ &\quad \left. - \frac{\psi_s^+(\tau) e^{-2i\pi k \tau}}{\eta(\tau)} K_{\overline{a+1},k}(\tau, \nu, -2\tau) + \frac{\psi_s^-(\tau) e^{-i\pi \frac{k}{2} \tau}}{\eta(\tau)} K_{\overline{a+1},k}(\tau, \nu, -\tau) \right] \\ &+ 2(-1)^r \sqrt{\frac{2}{p}} \frac{e^{i\pi k \frac{\nu^2}{2\tau}}}{\eta(\tau)} \sum_{\substack{s=1 \\ \bar{s}=a+k}}^{p-1} S_{rs}(\tau) \left[- \sum_{\substack{n=0 \\ \bar{n}=a+1}}^{2k-1} \left(\mathcal{B}_{n-k,s}(\tau) \theta_{n,k}(\tau, \nu) + \frac{1}{k} \mathcal{C}_{n-k,s}(\tau) \theta'_{n,k}(\tau, \nu) \right) \right. \\ &\quad \left. + \frac{\psi_s^+(\tau) e^{-i\pi \frac{k}{2} \tau}}{\eta(\tau)} K_{\overline{a+1},k}(\tau, \nu, -\tau) - \frac{\psi_s^-(\tau)}{\eta(\tau)} K_{\overline{a+1},k}(\tau, \nu, 0) \right] \\ &\quad + \frac{2\nu}{\sqrt{2p}} \frac{e^{i\pi k \frac{\nu^2}{2\tau}}}{\eta(\tau)} \sum_{\substack{s=1 \\ \bar{s}=a}}^{p-1} \sum_{\substack{n=0 \\ \bar{n}=a+1}}^{2k-1} \sin \frac{\pi rs}{p} \mathcal{C}_{n,s}(\tau) \theta_{n,k}(\tau, \nu). \end{aligned}$$

3.2.3. We now compare the two expressions for $\Omega_r^a(-\frac{1}{\tau}, \frac{\nu}{\tau})$, in **3.2.1** and **3.2.2**. The terms that explicitly involve ν already coincide in view of (1.10). The terms involving θ' are readily seen to coincide for the same reason (and because of (D.6)).

Next, comparing the terms involving K (or, equivalently, the residues of the two expressions for $\Omega_r^a(-\frac{1}{\tau}, \frac{\nu}{\tau})$), we recover the transformations of the $(p, 1)$ -model characters

$\psi_r^\pm(\tau)$ in (1.7)–(1.8) (this seems to be a remarkably complicated way to derive these simple formulas). But most importantly, some of the $K_{\overline{a+1},k}$ -terms contribute to $\theta_{n,k}$ -terms in accordance with **B.1.3**. Comparing the $\theta_{n,k}$ -terms then gives the relation

$$(3.1) \quad 2\sqrt{2k}\mathcal{B}_{m,r}\left(-\frac{1}{\tau}\right) = \sum_{n=0}^{2k-1} e^{i\pi\frac{mn}{k}} \mathbb{B}_{r,b,n}(\tau), \quad \begin{aligned} \overline{m} &= \overline{r+1}, \\ 0 &\leq m \leq 2k-1, \end{aligned}$$

where we temporarily use the notation

$$\begin{aligned} \mathbb{B}_{r,b,n}(\tau) &= -\sqrt{\frac{2}{p}} \frac{ir}{2p} \left[(-1)^r (1 + (-1)^{n+1+k}) \mathcal{B}_{-n,p}(\tau) + (1 + (-1)^{n+1}) \mathcal{B}_{n-k,p}(\tau) \right] \\ &\quad - 2\sqrt{\frac{2}{p}} \sum_{\substack{s=1 \\ \overline{s}=n+1}}^{p-1} S_{rs}(\tau) \mathcal{B}_{-n,s}(\tau) - 2\sqrt{\frac{2}{p}} (-1)^r \sum_{\substack{s=1 \\ \overline{s}=n+1+k}}^{p-1} S_{rs}(\tau) \mathcal{B}_{n-k,s}(\tau) \\ &\quad + \frac{i\psi_r^+\left(-\frac{1}{\tau}\right) e^{2i\pi\frac{k}{\tau}}}{\eta(\tau)} \sum_{\beta \in \{0,1\}} (-1)^{(r+1)\beta} \Phi(2k\tau, 2k - n\tau - \beta k\tau) \\ &\quad - \frac{i\psi_r^-\left(-\frac{1}{\tau}\right) e^{i\pi\frac{k}{\tau}}}{\eta(\tau)} \sum_{\beta \in \{0,1\}} (-1)^{(r+1)\beta} \Phi(2k\tau, k - n\tau - \beta k\tau). \end{aligned}$$

We also used (1.4) here.

It now follows from (1.3), (1.4), and (B.8) that $\mathbb{B}_{r,b,n+k}(\tau) = (-1)^{r+1} \mathbb{B}_{r,b,n}(\tau)$, and therefore Eq. (3.1) can be rewritten as⁴

$$\sqrt{2k}\mathcal{B}_{m,r}\left(-\frac{1}{\tau}\right) = \sum_{n=0}^{k-1} e^{i\pi\frac{mn}{k}} \mathbb{B}_{r,b,n}(\tau), \quad \begin{aligned} \overline{m} &= \overline{r+1}, \\ 0 &\leq m \leq 2k-1. \end{aligned}$$

But in the $\psi_r^\pm(-\frac{1}{\tau})$ -terms in this sum, we then have

$$\begin{aligned} \sum_{n=0}^{k-1} e^{i\pi\frac{mn}{k}} \frac{i\psi_r^+\left(-\frac{1}{\tau}\right) e^{2i\pi\frac{k}{\tau}}}{\eta(\tau)} \sum_{\beta \in \{0,1\}} (-1)^{(r+1)\beta} \Phi(2k\tau, 2k - n\tau - \beta k\tau) &= \\ = \sum_{n=0}^{2k-1} e^{i\pi\frac{mn}{k}} \frac{i\psi_r^+\left(-\frac{1}{\tau}\right) e^{2i\pi\frac{k}{\tau}}}{\eta(\tau)} \Phi(2k\tau, 2k - n\tau), \quad \overline{m} = \overline{r+1}, \end{aligned}$$

and subsequently using (B.11) and then (B.13), we continue this as

$$= \frac{i\psi_r^+\left(-\frac{1}{\tau}\right)}{\eta(\tau)} e^{i\pi\frac{(m+2k)^2}{2k\tau}} \Phi\left(\frac{\tau}{2k}, 1 + \frac{m}{k}\right) = \sqrt{2k} \frac{\psi_r^+\left(-\frac{1}{\tau}\right)}{\sqrt{-i\tau}\eta(\tau)} \Phi\left(-\frac{2k}{\tau}, -\frac{m}{\tau}\right).$$

Thus rewritten, this term (and the $\psi_r^-\left(-\frac{1}{\tau}\right)$ -term similarly) naturally combines with the left-hand side of (3.1) into

$$\widetilde{\mathcal{B}}_{m,r}(\tau) = \mathcal{B}_{m,r}(\tau) - \frac{\psi_r^+(\tau)}{\eta(\tau)} \Phi(2k\tau, m\tau) + \frac{\psi_r^-(\tau)}{\eta(\tau)} \Phi(2k\tau, (m-k)\tau)$$

⁴We simultaneously see that $0 = \sum_{n=0}^{2k-1} e^{i\pi\frac{mn}{k}} \mathbb{B}_{r,b,n}(\tau)$ for $\overline{m} = \overline{r}$, which also follows from comparison of the $\theta_{n,k}$ -terms above.

such that

$$\begin{aligned} \mathcal{B}_{m,r}^{\sim}\left(-\frac{1}{\tau}\right) &= \\ &= -\frac{1}{\sqrt{kp}} \sum_{n=0}^{k-1} e^{i\pi \frac{mn}{k}} \left(\frac{ir}{2p} \left[(-1)^r (1 + (-1)^{n+1+k}) \mathcal{B}_{-n,p}(\tau) + (1 + (-1)^{n+1}) \mathcal{B}_{n-k,p}(\tau) \right] \right. \\ &\quad \left. + 2 \sum_{\substack{s=1 \\ \bar{s}=n+1}}^{p-1} S_{rs}(\tau) \mathcal{B}_{-n,s}(\tau) + 2(-1)^r \sum_{\substack{s=1 \\ \bar{s}=n+1+k}}^{p-1} S_{rs}(\tau) \mathcal{B}_{n-k,s}(\tau) \right) \end{aligned}$$

for $\bar{m} = \overline{r+1}$ and $0 \leq m \leq 2k-1$. With (D.5) and after simple transformations, this can be conveniently rewritten as

$$\begin{aligned} \mathcal{B}_{m,r}^{\sim}\left(-\frac{1}{\tau}\right) &= \frac{(-1)^r}{\sqrt{kp}} \frac{ir}{2p} (1 + (-1)^{k+1}) \frac{\psi_p^+(\tau)}{2\eta(\tau)} \\ &\quad + \frac{1}{\sqrt{kp}} \sum_{\substack{s=1 \\ \bar{s}=1}}^{p-1} S_{rs}(\tau) \frac{\psi_s^+(\tau)}{\eta(\tau)} + \frac{(-1)^r}{\sqrt{kp}} \sum_{\substack{s=1 \\ \bar{s}=k+1}}^{p-1} S_{rs}(\tau) \frac{\psi_s^-(\tau)}{\eta(\tau)} \\ &\quad + \frac{(-1)^r}{\sqrt{kp}} \frac{2r}{p} \sum_{\substack{n=1 \\ \bar{n}=k+1}}^{k-1} \sin \frac{\pi mn}{k} \mathcal{B}_{-n,p}(\tau) - \frac{4i}{\sqrt{kp}} \sum_{n=1}^{k-1} \sum_{\substack{s=1 \\ \bar{n}+s=1}}^{p-1} \sin \frac{\pi mn}{k} S_{rs}(\tau) \mathcal{B}_{-n,s}(\tau), \end{aligned}$$

whence **1.3.3** is immediate.

3.3. $\mathcal{A}_{n,r}(-\frac{1}{\tau})$. A similar calculation of $\mathcal{A}_{n,r}(-\frac{1}{\tau})$ is straightforward in principle but rather bulky in practical terms. We begin with introducing the linear combinations of characters

$$(3.2) \quad \Xi_r^a(\tau, \nu) = \chi_r^+(\tau, \nu) + (-1)^a (\chi_r^-(\tau, \nu) + \frac{1}{4} \chi_{p-r}(\tau, \nu))$$

3.3.1. From **2.2.2** and (A.6)–(A.8), we calculate

$$\begin{aligned} \Xi_r^a\left(-\frac{1}{\tau}, \frac{\nu}{\tau}\right) &= \nu \Omega_r^a\left(-\frac{1}{\tau}, \frac{\nu}{\tau}\right) + \frac{2e^{i\pi \frac{kv^2}{2\tau}}}{\sqrt{2k}\eta(\tau)} \sum_{\substack{m=0 \\ \bar{m}=r+1}}^{2k-1} \sum_{\substack{n=0 \\ \bar{n}=a+1}}^{2k-1} e^{-i\pi \frac{mn}{k}} \left(\mathcal{A}_{m,r}\left(-\frac{1}{\tau}\right) \theta_{n,k}(\tau, \nu) \right. \\ &\quad \left. + \frac{2\tau}{k} \mathcal{B}_{m,r}\left(-\frac{1}{\tau}\right) \theta'_{n,k}(\tau, \nu) + \frac{\tau^2}{k^2} \mathcal{C}_{m,r}\left(-\frac{1}{\tau}\right) \theta''_{n,k}(\tau, \nu) + \left(\frac{\tau}{4i\pi k} - \frac{\nu^2}{4}\right) \mathcal{C}_{m,r}\left(-\frac{1}{\tau}\right) \theta_{n,k}(\tau, \nu) \right) \\ &\quad + \frac{\psi_r^+(-\frac{1}{\tau})}{i\tau\eta(\tau)^2} e^{2i\pi \frac{k}{\tau}} \left(\frac{2}{k} \mathbb{K}_{\frac{a+1}{r+1}}' \left(-\frac{1}{\tau}, \frac{\nu}{\tau}, \frac{2}{\tau}\right) - (2 + \nu) \mathbb{K}_{\frac{a+1}{r+1}} \left(-\frac{1}{\tau}, \frac{\nu}{\tau}, \frac{2}{\tau}\right) \right) \\ &\quad - \frac{\psi_r^-(-\frac{1}{\tau})}{i\tau\eta(\tau)^2} e^{i\pi \frac{k}{2\tau}} \left(\frac{2}{k} \mathbb{K}_{\frac{a+1}{r+1}}' \left(-\frac{1}{\tau}, \frac{\nu}{\tau}, \frac{1}{\tau}\right) - (1 + \nu) \mathbb{K}_{\frac{a+1}{r+1}} \left(-\frac{1}{\tau}, \frac{\nu}{\tau}, \frac{1}{\tau}\right) \right) \end{aligned}$$

We note that in the “ $\tau\nu\mu$ ” notation,

$$\mathcal{K}'_{\alpha,k}(\tau, \nu, \mu) = \frac{1}{2i\pi} \left(\frac{\partial}{\partial \nu} - \frac{\partial}{\partial \mu} \right) \mathcal{K}_{\alpha,k}(\tau, \nu, \mu).$$

In substituting the S -transformed \mathbb{K}_{r+1}^{a+1} functions here, we evaluate the relevant combinations $e^{i\pi k \frac{\mu^2}{2\tau}} \left(\frac{2}{k} \mathbb{K}_{r+1}^{a+1'} \left(-\frac{1}{\tau}, \frac{\nu}{\tau}, \frac{\mu}{\tau} \right) - (\nu + \mu) \mathbb{K}_{r+1}^{a+1} \left(-\frac{1}{\tau}, \frac{\nu}{\tau}, \frac{\mu}{\tau} \right) \right)$ at $\mu = 1$ and 2 using the identity

$$\begin{aligned} & \frac{1}{\tau} e^{i\pi k \frac{\mu^2}{2\tau}} \left(\frac{2}{k} \mathbb{K}_{r+1}^{a+1'} \left(-\frac{1}{\tau}, \frac{\nu}{\tau}, \frac{\mu}{\tau} \right) - (\mu + \nu) \mathbb{K}_{r+1}^{a+1} \left(\frac{1}{\tau}, \frac{\nu}{\tau}, \frac{\mu}{\tau} \right) \right) = \\ & = \tau e^{i\pi k \frac{\nu^2}{2\tau}} \left(\frac{2}{k} \mathbb{K}'_{a+1,k}(\tau, \nu, \mu) + (-1)^{r+1} e^{-i\pi k \frac{\tau}{2} + i\pi k \mu} \frac{2}{k} \mathbb{K}'_{a+1,k}(\tau, \nu, \mu - \tau) \right. \\ & \quad \left. - (-1)^{r+1} e^{-i\pi k \frac{\tau}{2} + i\pi k \mu} \mathbb{K}_{a+1,k}(\tau, \nu, \mu - \tau) \right) \\ & + \frac{2}{k} \tau e^{i\pi k \frac{\nu^2 + \mu^2}{2\tau}} \sum_{\beta \in \{0,1\}} \sum_{\substack{n=0 \\ \bar{n}=a+1}}^{2k-1} (-1)^{(r+1)\beta} \Phi(2k\tau, k\mu - n\tau - \beta k\tau) \theta'_{n,k}(\tau, \nu) \\ & - \frac{\tau}{i\pi k} \frac{\partial}{\partial \mu} \left(e^{i\pi k \frac{\nu^2 + \mu^2}{2\tau}} \sum_{\beta \in \{0,1\}} \sum_{\substack{n=0 \\ \bar{n}=a+1}}^{2k-1} (-1)^{(r+1)\beta} \Phi(2k\tau, k\mu - n\tau - \beta k\tau) \theta_{n,k}(\tau, \nu) \right), \end{aligned}$$

which readily follows from **B.3.1**. It is left to the reader to substitute the last formula (twice) in the above expression for $\Xi_r^a(-\frac{1}{\tau}, \frac{\nu}{\tau})$.

3.3.2. On the other hand, it follows from **C.2.3** that

$$\begin{aligned} \Xi_r^a(-\frac{1}{\tau}, \frac{\nu}{\tau}) &= \nu \Omega_r^a(-\frac{1}{\tau}, \frac{\nu}{\tau}) + \\ & + \sqrt{\frac{2}{p}} e^{i\pi k \frac{\nu^2}{2\tau}} \left(\tau(1 + (-1)^{k+a}) \frac{(-1)^{r} i r}{2p} \chi_p^+(\tau, \nu) + \tau(1 + (-1)^a) \frac{i r}{2p} \chi_p^-(\tau, \nu) \right. \\ & + 2 \sum_{\substack{s=1 \\ \bar{s}=a}}^{p-1} \tau S_{rs}(\tau) \chi_s^+(\tau, \nu) + 2(-1)^r \sum_{\substack{s=1 \\ \bar{s}=a+k}}^{p-1} \tau S_{rs}(\tau) \chi_s^-(\tau, \nu) \\ & + 2 \sum_{\substack{s=1 \\ \bar{s}=a}}^{p-1} \frac{i r}{p} \tau \frac{s}{2p} \cos \frac{\pi r s}{p} \chi_s(\tau, \nu) + 2 \sum_{\substack{s=1 \\ \bar{s}=a}}^{p-1} \left(\frac{\nu^2}{4} - \frac{s^2 \tau^2}{4p^2} \right) \sin \frac{\pi r s}{p} \chi_s(\tau, \nu) \\ & \quad \left. - \left(\frac{r^2}{2p^2} + \frac{\tau}{2i\pi p} \right) \sum_{\substack{s=1 \\ \bar{s}=a}}^{p-1} \sin \frac{\pi r s}{p} \chi_s(\tau, \nu) \right). \end{aligned}$$

We then use the decompositions for $\chi_s^\pm(\tau, \nu)$ (and $\omega_s^\pm(\tau, \nu)$ and $\chi_s(\tau, \nu)$) again, expressing χ_s^+ from **2.2.2** and χ_s^- from **2.3.3**. The substitution is totally straightforward, but the result is rather cumbersome, and we leave it to the reader to expand the last formula.

3.3.3. We next compare the two (rather cumbersome) expressions for $\Xi_r^a(-\frac{1}{\tau}, \frac{\nu}{\tau})$, resulting from **3.3.1** and **3.3.2**. The terms proportional to ν are already written as $\nu \Omega_r^a$ and therefore cancel. The terms proportional to ν^2 are readily seen to cancel due to the S -transformation properties of $\mathcal{C}_{m,r}$. The terms involving $\theta''_{n,k}$ cancel for the same reason.

Further, all terms involving $\theta'_{n,k}$ cancel due to the S -transformation properties of $\mathcal{B}_{m,r}$.⁵ After cancellations of the $K'_{a+1,k}$ and $K_{a+1,k}$, based on the identity

$$\begin{aligned} e^{-2i\pi k\tau} \left(\frac{2}{k} K'_{a+1,k}(\tau, \nu, -2\tau) - 2K_{a+1,k}(\tau, \nu, -2\tau) \right) = \\ = \frac{2}{k} K'_{a+1,k}(\tau, \nu, 0) + \sum_{\substack{n=0 \\ \bar{n}=a+1}}^{2k-1} \left(\frac{n}{k} e^{-i\pi\tau\frac{n^2}{2k}} \theta_{n,k}(\tau, \nu) - \frac{2}{k} e^{-i\pi\tau\frac{n^2}{2k}} \theta'_{n,k}(\tau, \nu) \right) \end{aligned}$$

following from **B.1.3**, we obtain

$$2\sqrt{2k}\mathcal{A}_{m,r}\left(-\frac{1}{\tau}\right) = \sum_{n=0}^{2k-1} e^{i\pi\frac{mn}{k}} \mathbb{A}_{r,b,n}(\tau), \quad \bar{m} = \overline{r+1}, \quad 0 \leq m \leq 2k-1,$$

where we introduce the temporary notation

$$\begin{aligned} \mathbb{A}_{r,b,n}(\tau) = & \sqrt{\frac{2}{p}} \left(\tau(1 + (-1)^{k+n+1}) \frac{(-1)^r i r}{2p} \mathcal{A}_{-n,p}(\tau) - \tau(1 + (-1)^{n+1}) \frac{i r}{2p} \mathcal{A}_{n-k,p}(\tau) \right. \\ & + 2 \sum_{\substack{s=1 \\ \bar{s}=n+1}}^{p-1} \tau S_{rs}(\tau) \mathcal{A}_{-n,s}(\tau) - 2(-1)^r \sum_{\substack{s=1 \\ \bar{s}=n+k+1}}^{p-1} \tau S_{rs}(\tau) \mathcal{A}_{n-k,s}(\tau) \\ & + \sum_{\substack{s=1 \\ \bar{s}=n+1}}^{p-1} \frac{i r}{p} \tau \frac{p-2s}{2p} \cos \frac{\pi r s}{p} \mathcal{C}_{n,s}(\tau) + \sum_{\substack{s=1 \\ \bar{s}=n+1}}^{p-1} \left(\frac{(s^2 - ps)\tau^2}{2p^2} + \frac{r^2}{2p^2} - \frac{\tau}{i\pi p k} \right) \sin \frac{\pi r s}{p} \mathcal{C}_{n,s}(\tau) \Bigg) \\ & + \frac{\psi_r^+(-\frac{1}{\tau})}{i\eta(\tau)} \frac{\tau}{i\pi k} \frac{\partial}{\partial \mu} \sum_{\beta \in \{0,1\}} (-1)^{(r+1)\beta} e^{i\pi k \frac{\mu^2}{2\tau}} \Phi(2k\tau, k\mu - n\tau - \beta k\tau) \Big|_{\mu=2} \\ & - \frac{\psi_r^-(-\frac{1}{\tau})}{i\eta(\tau)} \frac{\tau}{i\pi k} \frac{\partial}{\partial \mu} \sum_{\beta \in \{0,1\}} (-1)^{(r+1)\beta} e^{i\pi k \frac{\mu^2}{2\tau}} \Phi(2k\tau, k\mu - n\tau - \beta k\tau) \Big|_{\mu=1}. \end{aligned}$$

We also used (1.14) here. It now follows from (1.13), (1.14), and (B.8) that $\mathbb{A}_{r,b,n+k}(\tau) = (-1)^{r+1} \mathbb{A}_{r,b,n}(\tau)$, and therefore

$$\sum_{n=0}^{2k-1} e^{i\pi\frac{mn}{k}} \mathbb{A}_{r,b,n}(\tau) = 2 \sum_{n=0}^{k-1} e^{i\pi\frac{mn}{k}} \mathbb{A}_{r,b,n}(\tau).$$

But in the $\psi_r^\pm(-\frac{1}{\tau})$ -terms in the sum in the right-hand side, we then have (see **B.2** and **B.3.2**)

$$\sum_{n=0}^{k-1} e^{i\pi\frac{mn}{k}} \frac{\psi_r^\pm(-\frac{1}{\tau})}{i\eta(\tau)} \frac{\tau}{i\pi k} \frac{\partial}{\partial \mu} \sum_{\beta \in \{0,1\}} (-1)^{(r+1)\beta} e^{i\pi k \frac{\mu^2}{2\tau}} \Phi(2k\tau, k\mu - n\tau - \beta k\tau) =$$

⁵The calculation with χ_r^\pm alone establishes the transformations of $\mathcal{C}_{m,r}$ and $\mathcal{B}_{m,r}$ as well as of $\mathcal{A}_{m,r}$, but we prefer to have the formula for $\mathcal{B}_{m,r}(-\frac{1}{\tau})$ already derived and to use it in the (rather tedious) χ_r^\pm -calculation only for control.

$$\begin{aligned}
&= \frac{\tau}{i\pi k} \frac{\partial}{\partial \mu} \frac{\psi_r^\pm(-\frac{1}{\tau})}{i\eta(\tau)} e^{i\pi k \frac{\mu^2}{2\tau}} \sum_{n=0}^{2k-1} e^{i\pi \frac{mn}{k}} \Phi(2k\tau, k\mu - n\tau) = \\
&= \frac{\tau}{i\pi k} \frac{\partial}{\partial \mu} \frac{\psi_r^\pm(-\frac{1}{\tau})}{i\eta(\tau)} e^{i\pi \frac{(m+k\mu)^2}{2k\tau}} \Phi\left(\frac{\tau}{2k}, \frac{\mu}{2} + \frac{m}{2k}\right) = \\
&= \sqrt{2k} \frac{2\psi_r^\pm(-\frac{1}{\tau})}{\sqrt{-i\tau\eta(\tau)}} \Phi'\left(-\frac{2k}{\tau}, -\frac{m+(\mu-2)k}{\tau}\right).
\end{aligned}$$

Hence, defining $A_{m,r}(\tau)$ as in (1.16), we obtain the S -transform formula in **1.3.4**.

4. CONCLUSIONS

The higher string functions $\mathcal{A}_{n,r}(\tau)$ and $\mathcal{B}_{n,r}(\tau)$ are not “arbitrary” analogues of the $\mathcal{C}_{n,r}(\tau)$: there is an underlying representation-theory picture described in [2]. The associated conformal field theory construction (the W -algebra in [2]) may then be considered the rationale for the higher string functions to have interesting modular properties.

A “feedback” of modular transformations to conformal field theory is that they come to play the role of a strong consistency check (e.g., for the field content) whenever representation-theory details are not known, as is the case with the logarithmic extension of the parafermion theory, where only the characters are available but the field-theory picture is presently obscure. As in the previously known cases of logarithmic $(p, 1)$ and (p, p') models [23, 32], the degree of the polynomials in τ occurring in modular transformations may be expected to correlate with the Jordan cell sizes in indecomposable representations of the corresponding extended algebra, but the representation-theory interpretation of the occurrences of the “ Φ -constants” $\Phi(2k\tau, m\tau)$ (times the $(p, 1)$ -model characters) is a challenging problem.

Modular transformations are related to fusion, via the Verlinde formula in rational conformal field theories [40] and via its generalizations in logarithmic theories [22, 41, 17]; whether the modular transformations derived in this paper lead to any reasonable non-semisimple fusion algebra remains an interesting problem.

The celebrated form of $\mathcal{C}_{n,r}(q)$ first found in [31] has been the subject of considerable attention since then; it would be interesting to find an extension of such representations to the “logarithmically extended parafermionic characters” $\mathcal{A}_{m,r}(q)$ and $\mathcal{B}_{m,r}(q)$. Different “fermionic-type” character formulas may also be mentioned in this connection (see [42] and the numerous subsequent papers, in particular, e.g., [43, 44, 45] and the references therein). Their counterparts for $\mathcal{A}_{m,r}(q)$ and $\mathcal{B}_{m,r}(q)$ may also be interesting.

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APPENDIX A. THETA-FUNCTION CONVENTIONS

The level- \varkappa theta-functions are defined as

$$(A.1) \quad \theta_{r,\varkappa}(q, z) = \sum_{\iota \in \mathbb{Z} + \frac{r}{2\varkappa}} q^{\varkappa \iota^2} z^{\varkappa \iota}.$$

We set

$$(A.2) \quad \theta'_{r,\varkappa}(q, z) = z \frac{\partial}{\partial z} \theta_{r,\varkappa}(q, z), \quad \theta''_{r,\varkappa}(q, z) = \left(z \frac{\partial}{\partial z} \right)^2 \theta_{r,\varkappa}(q, z)$$

and

$$(A.3) \quad \theta'_{r,\varkappa}(q) = \theta'_{r,\varkappa}(q, z) \Big|_{z=1}.$$

The quasiperiodicity properties of theta-functions are expressed as

$$(A.4) \quad \theta_{r,\varkappa}(q, zq^n) = q^{-\varkappa \frac{n^2}{4}} z^{-\varkappa \frac{n}{2}} \theta_{r+\varkappa n, \varkappa}(q, z),$$

with $\theta_{r+\varkappa n, \varkappa}(q, z) = \theta_{r,\varkappa}(q, z)$ for even n .

The modular T -transform of theta-functions is

$$(A.5) \quad \theta_{r,\varkappa}(\tau + 1, \nu) = e^{i\pi \frac{\tau^2}{2\varkappa}} \theta_{r,\varkappa}(\tau, \nu)$$

and the S -transform is

$$(A.6) \quad \theta_{r,\varkappa}\left(-\frac{1}{\tau}, \frac{\nu}{\tau}\right) = e^{i\pi \frac{\varkappa \nu^2}{2\tau}} \sqrt{\frac{-i\tau}{2\varkappa}} \sum_{s=0}^{2\varkappa-1} e^{-i\pi \frac{rs}{\varkappa}} \theta_{s,\varkappa}(\tau, \nu).$$

Therefore,

$$(A.7) \quad \theta'_{r,\varkappa}\left(-\frac{1}{\tau}, \frac{\nu}{\tau}\right) = e^{i\pi \frac{\varkappa \nu^2}{2\tau}} \sqrt{\frac{-i\tau}{2\varkappa}} \sum_{s=0}^{2\varkappa-1} e^{-i\pi \frac{rs}{\varkappa}} \left(\tau \theta'_{s,\varkappa}(\tau, \nu) + \frac{\varkappa \nu}{2} \theta_{s,\varkappa}(\tau, \nu) \right),$$

$$(A.8) \quad \theta''_{r,\varkappa}\left(-\frac{1}{\tau}, \frac{\nu}{\tau}\right) = e^{i\pi \frac{\varkappa \nu^2}{2\tau}} \sqrt{\frac{-i\tau}{2\varkappa}} \sum_{s=0}^{2\varkappa-1} e^{-i\pi \frac{rs}{\varkappa}} \left(\tau^2 \theta''_{s,\varkappa}(\tau, \nu) + \varkappa \nu \tau \theta'_{s,\varkappa}(\tau, \nu) + \left(\frac{\varkappa^2 \nu^2}{4} + \frac{\varkappa \tau}{4i\pi} \right) \theta_{s,\varkappa}(\tau, \nu) \right).$$

The price paid for abusing notation is that $\theta'_{r,\varkappa}(\tau, \nu) = \frac{1}{2i\pi} \frac{\partial}{\partial \nu} \theta_{r,\varkappa}(\tau, \nu)$.

We also use the Jacobi theta-functions

$$(A.9) \quad \vartheta_{1,1}(q, z) = \sum_{m \in \mathbb{Z}} q^{\frac{1}{2}(m^2-m)} (-z)^{-m} = \prod_{m \geq 0} (1 - z^{-1} q^m) \prod_{m \geq 1} (1 - z q^m) \prod_{m \geq 1} (1 - q^m),$$

$$(A.10) \quad \vartheta(q, z) = \sum_{m \in \mathbb{Z}} q^{\frac{m^2}{2}} z^m$$

related to (A.1) as

$$\theta_{r,\varkappa}(q, z) = z^{\frac{r}{2}} q^{\frac{r^2}{4\varkappa}} \vartheta(q^{2\varkappa}, z^{\varkappa} q^r).$$

For the function

$$(A.11) \quad \Omega(q, z) = q^{\frac{1}{8}} z^{\frac{1}{2}} \vartheta_{1,1}(q, z),$$

we then have the S -transformation formula

$$\Omega\left(-\frac{1}{\tau}, \frac{\nu}{\tau}\right) = -i\sqrt{-i\tau} e^{i\pi \frac{\nu^2}{2\tau}} \Omega(\tau, \nu).$$

The eta function

$$(A.12) \quad \eta(q) = q^{\frac{1}{24}} \prod_{m=1}^{\infty} (1 - q^m)$$

transforms as

$$(A.13) \quad \eta(\tau + 1) = e^{\frac{i\pi}{12}} \eta(\tau), \quad \eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau} \eta(\tau).$$

APPENDIX B. PROPERTIES OF THE APPELL FUNCTIONS [1]

B.1. Definition and simple properties.

B.1.1. For $k \in \mathbb{N}$, the level- k Appell function is defined as [1]

$$(B.1) \quad \mathcal{K}_k(q, x, y) = \sum_{m \in \mathbb{Z}} \frac{q^{\frac{m^2 k}{2}} x^{mk}}{1 - xyq^m},$$

It has a number of properties that nontrivially generalize the theta-function properties, the crucial ones being the “open quasiperiodicity”

$$\mathcal{K}_k(q^2, xq^{-\frac{2n}{k}}, yq^{\frac{2n}{k}}) = (xy)^n \mathcal{K}_k(q^2, x, y) + \sum_{r=1}^n (xy)^{n-r} x^r q^{-\frac{r^2}{k}} \theta_{-2r,k}(q, x), \quad n \in \mathbb{N}$$

(where it is worth noting that the x and y variables separate in the extra terms), and the modular transformation properties, Eq. (1.5) in particular (where, remarkably, ν and μ also separate in the extra terms in the right-hand side).

B.1.2. In this paper, we need a version of \mathcal{K}_k with the “period” q^2 and with characteristics. We define

$$(B.2) \quad \mathcal{K}_{\alpha,k}(q, x, y) = q^{-\frac{\alpha^2}{4k}} y^{\frac{\alpha}{2}} \sum_{m \in \mathbb{Z} + \frac{\alpha}{2k}} \frac{q^{km^2} x^{km}}{1 - xyq^{2m - \frac{\alpha}{k}}} = (xy)^{\frac{\alpha}{2}} \mathcal{K}_k(q^2, xq^{\frac{\alpha}{k}}, yq^{-\frac{\alpha}{k}}).$$

The open quasiperiodicity relation above implies that

$$(B.3) \quad \mathcal{K}_{\alpha+2n,k}(q, x, y) = \mathcal{K}_{\alpha,k}(q, x, y) - \sum_{\substack{m=0 \\ \overline{m}=\alpha}}^{2n-1} q^{-\frac{m^2}{4k}} y^{\frac{m}{2}} \theta_{m,k}(q, x),$$

and therefore the characteristic α in (B.2) can be reduced modulo 2 at the expense of theta functions.

B.1.3. Open quasiperiodicity in the third argument. It follows from the formulas in [1] (or can be easily derived from the definition) that $K_{\alpha,k}$ satisfies an open quasiperiodicity property with respect to the third argument:

$$K_{\alpha,k}(q, x, yq^{-2n}) = q^{kn^2} y^{-kn} \left(K_{\alpha,k}(q, x, y) - \sum_{b=0}^{nk-1} q^{-\frac{1}{4k}(\alpha+2b)^2} y^{\frac{\alpha}{2}+b} \theta_{\alpha+2b,k}(q, x) \right)$$

for $n \in \mathbb{N}$. In the text, we use this formula in the form

$$K_{\frac{a+1}{2},k}(\tau, \nu, \mu-2\tau) = e^{2i\pi k\tau-2i\pi k\mu} \left(K_{\frac{a+1}{2},k}(\tau, \nu, \mu) - \sum_{\substack{n=0 \\ \bar{n}=a+1}}^{2k-1} e^{-i\pi\tau\frac{n^2}{2k}+i\pi\mu n} \theta_{n,k}(\tau, \nu) \right).$$

B.1.4. Period-changing formula. We recall the elementary theta-function identity

$$(B.4) \quad \vartheta(q, z) = \sum_{s=0}^{u-1} q^{\frac{s^2}{2}} z^s \vartheta(q^{u^2}, z^u q^{us}), \quad u \in \mathbb{N}.$$

Similarly to (B.4), we relate $\mathcal{K}_k(q^2, x, y)$ to suchlike functions with the “period” q^{u^2} for any $u \in \mathbb{N}$ as

$$\mathcal{K}_k(q^2, x, y) = \sum_{a=0}^{u-1} \sum_{b=0}^{u-1} q^{-ka^2} y^{-ka} K_{\frac{2b}{u}+\frac{2ka}{u},k}(q^{u^2}, x^u, y^u q^{2ua}).$$

This formula may not be very useful for general u because of the fractional characteristics in the right-hand side, but for $u = 2$ it takes the simple form

$$(B.5) \quad \mathcal{K}_k(q^{\frac{1}{2}}, x^{\frac{1}{2}}, y^{\frac{1}{2}}) = \sum_{\gamma, \beta \in \{0,1\}} q^{-\frac{k\gamma^2}{4}} y^{-\frac{k\gamma}{2}} K_{\beta+k\gamma,k}(q, x, yq^\gamma) = \sum_{\beta \in \{0,1\}} \mathbb{K}_\beta^0(q, x, y).$$

B.2. The Φ function. The Φ function defined in (1.1) can be equivalently written as the b -cycle integral [1, Eq. (A.5)]

$$(B.6) \quad \sqrt{-i\tau} \Phi(\tau, \mu) = i \int_0^\tau d\lambda e^{i\pi\frac{\lambda^2-2\lambda\mu}{\tau}} \mathcal{K}_1(\tau, \lambda + \varepsilon - \mu, \mu)$$

(where an infinitesimal positive real ε specifies the prescription to bypass the singularities). This close relative of the Mordell integral has a number of useful properties [1]. First, it is “open-double-quasiperiodic” under shifts of the argument by $n + m\tau$, $m, n \in \mathbb{Z}$:

$$(B.7) \quad \Phi(\tau, \mu+n) = e^{-i\pi\frac{n^2}{\tau}-2i\pi n\frac{\mu}{\tau}} \Phi(\tau, \mu) + \frac{i}{\sqrt{-i\tau}} \sum_{j=1}^n e^{i\pi\frac{j(j-2n)}{\tau}-2i\pi j\frac{\mu}{\tau}}, \quad n \in \mathbb{N},$$

$$(B.8) \quad \Phi(\tau, \mu-m\tau) = \Phi(\tau, \mu) + \sum_{j=0}^{m-1} e^{-i\pi\frac{(\mu-j\tau)^2}{\tau}}, \quad m \in \mathbb{N}.$$

Together with the “reflection” property

$$(B.9) \quad \Phi(\tau, -\mu) = \frac{-i}{\sqrt{-i\tau}} - e^{-i\pi\frac{\mu^2}{\tau}} - \Phi(\tau, \mu),$$

this allows evaluating $\Phi(\tau, \cdot)$ at some (not all) of the half-period points:

$$\Phi(\tau, \frac{n}{2}) = -\frac{1}{2} e^{-i\pi \frac{n^2}{4\tau}} + \frac{i}{2\sqrt{-i\tau}} \sum_{j=1}^{n-1} e^{i\pi \frac{j(j-n)}{\tau}}, \quad n \geq 1,$$

and

$$\Phi(\tau, \frac{m\tau}{2}) = -\frac{i}{2\sqrt{-i\tau}} - \frac{1}{2} \sum_{j=0}^m e^{-i\pi \tau \frac{(m-2j)^2}{4}}, \quad m \geq 0,$$

whence $\Phi(\tau, \frac{n}{2} + \frac{m}{2} \tau)$ with even nm follow via the open quasiperiodicity formulas above (formulas (B.7)–(B.9) do not allow finding $\Phi(\tau, \frac{n}{2} + \frac{m}{2} \tau)$ with n and m both odd).

Next, a simple “scaling law” is given by

$$(B.10) \quad \Phi(\tau, \mu) = \sum_{b=0}^{u-1} \Phi(u^2 \tau, u\mu - bu\tau), \quad u \in \mathbb{N}.$$

In the case of “scaling” with an even factor, we have the formula [1]

$$(B.11) \quad \sum_{n=0}^{2k-1} e^{i\pi \frac{mn}{k}} \Phi(2k\tau, 2k\mu - n\tau) = e^{i\pi \frac{[m]_{2k}^2}{2k\tau} + 2i\pi \frac{\mu}{\tau} [m]_{2k}} \Phi(\frac{\tau}{2k}, \mu + \frac{[m]_{2k}}{2k}).$$

Modular properties of Φ are considered after those of the Appell functions, in **B.3.2**.

B.3. Modular transformation properties. The S -transformation of the Appell functions $K_{\alpha,k}(2\tau, \nu, \mu)$ can be derived from (1.5). We need a version of the S -transformation formula for the functions $\mathbb{K}_{\alpha}^a(\tau, \nu, \mu)$ introduced in **3.2**. The following lemma plays a crucial role in the calculations in Sec. 3.⁶

B.3.1. Lemma. *We have the S -transform formula*

$$\begin{aligned} \mathbb{K}_{\tau}^a(-\frac{1}{\tau}, \frac{\nu}{\tau}, \frac{\mu}{\tau}) &= \tau e^{i\pi k \frac{\nu^2 - \mu^2}{2\tau}} \left(\mathbb{K}_{\bar{a},k}(\tau, \nu, \mu) + (-1)^r e^{i\pi k \mu - i\pi k \frac{\tau}{2}} \mathbb{K}_{\bar{a},k}(\tau, \nu, \mu - \tau) \right) \\ &\quad + \tau e^{i\pi k \frac{\nu^2}{2\tau}} \sum_{\beta \in \{0,1\}} \sum_{\substack{n=0 \\ \bar{n}=\bar{a}}}^{2k-1} (-1)^{r\beta} \Phi(2k\tau, k\mu - n\tau - \beta k\tau) \theta_{n,k}(\tau, \nu). \end{aligned}$$

Proof. Several properties of the Appell functions and of the Φ function are used here. First, with (B.4) and (B.5),⁷ it readily follows from (1.5) that

$$\begin{aligned} K_{\alpha,k}(-\frac{1}{\tau}, \frac{\nu}{\tau}, \frac{\mu}{\tau}) &= \frac{\tau}{2} e^{i\pi k \frac{\nu^2 - \mu^2}{2\tau}} \sum_{\beta \in \{0,1\}} \mathbb{K}_{\beta}^{\alpha}(\tau, \nu, \mu) \\ &\quad + \frac{\tau}{2} e^{i\pi k \frac{\nu^2}{2\tau}} \sum_{b=0}^{k-1} \sum_{\gamma=0}^1 e^{i\pi \frac{\alpha^2}{2k\tau} - i\pi \frac{\alpha b}{k} + i\pi \frac{\mu}{\tau} \alpha} (-1)^{\alpha\gamma} \Phi(\frac{k\tau}{2}, \frac{k\mu + \alpha - b\tau}{2}) \theta_{b+k\gamma,k}(\tau, \nu). \end{aligned}$$

⁶The lemma also explains the usefulness of the \mathbb{K}_{α}^a functions: the sign factor $(-1)^a$ in the left-hand side of the formula in the lemma becomes the characteristic, reduced to $\{0, 1\}$, in the right-hand side.

⁷And the easily verified property $\mathcal{K}_k(\tau, \nu + \frac{m}{k}, \mu - \frac{m}{k}) = \mathcal{K}_k(\tau, \nu, \mu)$, $m \in \mathbb{Z}$.

We next rewrite this for the characteristic α replaced with $\alpha + \bar{r}$ and use (B.10) with $u = 2$:

$$\begin{aligned} \mathbb{K}_{\alpha+\bar{r},k}\left(-\frac{1}{\tau}, \frac{\nu}{\tau}, \frac{\mu}{\tau}\right) &= \frac{\tau}{2} e^{i\pi k \frac{\nu^2 - \mu^2}{2\tau}} \sum_{\beta \in \{0,1\}} \mathbb{K}_{\beta}^{\overline{\alpha+\bar{r}}}(\tau, \nu, \mu) \\ &\quad + \frac{\tau}{2} e^{i\pi k \frac{\nu^2}{2\tau} + i\pi \frac{\alpha\mu}{\tau} + i\pi \frac{\alpha^2}{2k\tau}} \sum_{\beta, \gamma \in \{0,1\}} \sum_{b=0}^{k-1} (-1)^{\alpha\gamma + r\gamma + r\beta} e^{-i\pi \frac{\alpha b}{k}} \times \\ &\quad \times \Phi(2k\tau, k\mu + \alpha - b\tau - \beta k\tau) \theta_{b+k\gamma, k}(\tau, \nu). \end{aligned}$$

It then follows that

$$\begin{aligned} \text{(B.12)} \quad \mathbb{K}_{\alpha+\bar{r}}^a\left(-\frac{1}{\tau}, \frac{\nu}{\tau}, \frac{\mu}{\tau}\right) &= \\ &= \tau e^{i\pi k \frac{\nu^2 - \mu^2}{2\tau}} \left(\mathbb{K}_{\bar{a}, k}(\tau, \nu, \mu) + (-1)^{\alpha+r} e^{-i\pi k \frac{\tau}{2} - i\pi k \mu} \mathbb{K}_{\overline{a+k+k}, k}(\tau, \nu, \mu + \tau) \right) \\ &\quad + \tau e^{i\pi k \frac{\nu^2}{2\tau} + i\pi \frac{\alpha\mu}{\tau} + i\pi \frac{\alpha^2}{2k\tau}} X_{\alpha, \bar{r}}^a(\tau, \nu, \mu), \end{aligned}$$

where we introduce the temporary notation

$$\begin{aligned} X_{\alpha, \bar{r}}^a(\tau, \nu, \mu) &= \frac{1}{2} \sum_{\beta, \gamma \in \{0,1\}} \sum_{b=0}^{k-1} (1 + (-1)^{a+b+k\gamma}) e^{-i\pi \frac{\alpha b}{k}} \\ &\quad \times (-1)^{\alpha\gamma + r\gamma + r\beta} \Phi(2k\tau, k\mu + \alpha - b\tau - \beta k\tau) \theta_{b+k\gamma, k}(\tau, \nu). \end{aligned}$$

We next observe that by virtue of (B.8),

$$\begin{aligned} X_{\alpha, \bar{r}}^a(\tau, \nu, \mu) &+ (-1)^{r+\alpha} \sum_{b=0}^{k-1} (1 + (-1)^{a+b+k}) e^{-i\pi \frac{\alpha b}{k}} e^{-i\pi \frac{(k\mu + \alpha - b\tau)^2}{2k\tau}} \theta_{b+k\gamma, k}(\tau, \nu) \\ &= \frac{1}{2} \sum_{\beta, \gamma \in \{0,1\}} \sum_{b=0}^{k-1} (1 + (-1)^{a+b+k\gamma}) e^{-i\pi \frac{\alpha b}{k}} (-1)^{\alpha\gamma + r\beta} \\ &\quad \times \Phi(2k\tau, k\mu + \alpha - (b+k\gamma)\tau - \beta k\tau) \theta_{b+k\gamma, k}(\tau, \nu) \end{aligned}$$

and it is easy to see that the equality continues as

$$= \sum_{\beta \in \{0,1\}} \sum_{\substack{n=0 \\ \bar{n}=a}}^{2k-1} e^{-i\pi \frac{\alpha n}{k}} (-1)^{r\beta} \Phi(2k\tau, k\mu + \alpha - n\tau - \beta k\tau) \theta_{n, k}(\tau, \nu).$$

Substituting the $X_{\alpha, \bar{r}}^a(\tau, \nu, \mu)$ thus expressed in (B.12), we also use **B.1.3** to replace $\mathbb{K}_{\overline{a+k+k}, k}(\tau, \nu, \mu + \tau)$ with $\mathbb{K}_{\overline{a+k+k}, k}(\tau, \nu, \mu - \tau)$. After some cancellations, this gives

$$\begin{aligned} \mathbb{K}_{\alpha+\bar{r}}^a\left(-\frac{1}{\tau}, \frac{\nu}{\tau}, \frac{\mu}{\tau}\right) &= \tau e^{i\pi k \frac{\nu^2 - \mu^2}{2\tau}} \left(\mathbb{K}_{\bar{a}, k}(\tau, \nu, \mu) + (-1)^{\alpha+r} e^{i\pi k \mu - i\pi k \frac{\tau}{2}} \mathbb{K}_{\overline{a+k+k}, k}(\tau, \nu, \mu - \tau) \right) \\ &\quad + \tau e^{i\pi k \frac{\nu^2}{2\tau} + i\pi \frac{\alpha\mu}{\tau} + i\pi \frac{\alpha^2}{2k\tau}} \sum_{\beta \in \{0,1\}} \sum_{\substack{n=0 \\ \bar{n}=\bar{a}}}^{2k-1} (-1)^{r\beta} e^{-i\pi \frac{\alpha n}{k}} \Phi(2k\tau, k\mu + \alpha - n\tau - \beta k\tau) \theta_{n, k}(\tau, \nu) \end{aligned}$$

$$+ \tau e^{i\pi k \frac{\nu^2 - \mu^2}{2\tau}} (-1)^{\alpha+r} \sum_{\substack{n=k \\ \bar{n}=a+k}}^{2k-1} e^{-i\pi \frac{\tau n^2}{2k} + i\pi \mu n} \theta_{n-k,k}(\tau, \nu).$$

We finally apply (B.3) to $\mathbb{K}_{\overline{a+k+k},k}(\tau, \nu, \mu - \tau)$ in the last formula. Because $\overline{a+k+k} = \bar{a}$, we have $\overline{a+k+k} = \bar{a} + 2\ell$ with an integer ℓ , and therefore (B.3) is indeed applicable, with $\ell = \frac{1}{2}(k + \bar{k}) - 1$ if $\bar{a} = \bar{k} = 1$ and $\ell = \frac{1}{2}(k + \bar{k})$ otherwise. This gives

$$\begin{aligned} \mathbb{K}_{\alpha+\bar{r}}^a\left(-\frac{1}{\tau}, \frac{\nu}{\tau}, \frac{\mu}{\tau}\right) &= \tau e^{i\pi k \frac{\nu^2 - \mu^2}{2\tau}} \left(\mathbb{K}_{\bar{a},k}(\tau, \nu, \mu) + (-1)^{\alpha+r} e^{i\pi k \mu - i\pi k \frac{\tau}{2}} \mathbb{K}_{\bar{a},k}(\tau, \nu, \mu - \tau) \right) \\ &+ \tau e^{i\pi k \frac{\nu^2}{2\tau} + i\pi \frac{\alpha\mu}{\tau} + i\pi \frac{\alpha^2}{2k\tau}} \sum_{\beta \in \{0,1\}} \sum_{\substack{n=0 \\ \bar{n}=\bar{a}}}^{2k-1} (-1)^{r\beta} e^{-i\pi \frac{\alpha n}{k}} \Phi(2k\tau, k\mu + \alpha - n\tau - \beta k\tau) \theta_{n,k}(\tau, \nu), \end{aligned}$$

and the identity in the lemma is just the $\alpha = 0$ case of this. \square

B.3.2. We finally quote the S -transformation formula for the Φ function [1]:

$$(B.13) \quad \Phi\left(-\frac{1}{\tau}, \frac{\mu}{\tau}\right) = i\sqrt{-i\tau} e^{i\pi \frac{(\mu-1)^2}{\tau}} \Phi(\tau, 1 - \mu).$$

B.3.3. We note that all the properties of Φ in **B.2** can be derived directly from the definition as well as from the mere appearance of Φ in the S -transformation formula (1.5) (and from the properties of the Appell functions). In particular, (B.13) follows from $S^4 = 1$ evaluated on the Appell functions (see [1] for the details). In application to the higher string functions in this paper, it may be useful to formulate a more specific identity that “ensures” the relation $S^4 = 1$ evaluated, e.g., on $\mathcal{B}_{m,r}$. It follows from the S -dual relation to (B.11),

$$\frac{1}{2k} \sum_{n=0}^{2k-1} e^{2i\pi \frac{n\mu}{\tau} - i\pi \frac{mn}{k} + i\pi \frac{n^2}{2k\tau}} \Phi\left(\frac{\tau}{2k}, \mu + \frac{n}{2k}\right) = \Phi(2k\tau, 2k\mu - [m]_{2k}\tau),$$

and other properties of Φ in **B.2**: for $1 \leq m \leq k-1$ and $k+1 \leq m \leq 2k-1$, we have

$$2i \sum_{n=1}^{k-1} \sin \frac{\pi mn}{k} e^{i\pi \frac{n^2}{2k\tau}} \Phi\left(\frac{\tau}{2k}, \frac{n}{2k}\right) = -\frac{1 - (-1)^m}{2} i \cot \frac{\pi m}{2k} - \frac{i\sqrt{2k}}{2\sqrt{-i\tau}} - 2k\Phi(2k\tau, -m\tau).$$

APPENDIX C. $W(k)$ CHARACTERS IN THE LOGARITHMIC $\widehat{s\ell}(2)_k$ MODEL [2]

We here recall the spectral-flow and modular properties of the characters of the W -algebra constructed in [2].

C.1. Spectral-flow properties. Spectral flow automorphisms act on the character of any $\widehat{s\ell}(2)_k$ -module \mathcal{L} as [46, 2]

$$(C.1) \quad \chi_{;\theta}^{\mathcal{L}}(q, z) = q^{\frac{k}{4}\theta^2} z^{-\frac{k}{2}\theta} \chi^{\mathcal{L}}(q, zq^{-\theta}).$$

The integrable representation characters are well-known to be periodic under the spectral flow, $\chi_{r;1}(q, z) = \chi_{p-r}(q, z)$, and therefore $\chi_{r;2}(q, z) = \chi_r(q, z)$.

Clearly, the rule in (C.1) also applies to the characters of the extended algebra $W(k)$ of the logarithmically extended $\widehat{sl}(2)_k$ model. For χ_r^\pm in (2.4) and (2.5), calculation then shows that

$$(C.2) \quad \chi_{r;1}^+(q, z) = -\chi_r^-(q, z) - \omega_r^-(q, z) - \frac{1}{2}\chi_{p-r}(q, z),$$

$$(C.3) \quad \chi_{r;1}^-(q, z) = -\chi_r^+(q, z) - \omega_r^+(q, z)$$

for $1 \leq r \leq p-1$ (with $p = k+2$), and, similarly,

$$(C.4) \quad \chi_{p;1}^+(q, z) = -\chi_p^-(q, z) - \omega_p^-(q, z),$$

$$(C.5) \quad \chi_{p;1}^-(q, z) = -\chi_p^+(q, z) - \omega_p^+(q, z),$$

with the ω_r^\pm given in 2.2.1. On these, the spectral flow action as in (C.1) is in turn readily evaluated as

$$\begin{aligned} \omega_{r;1}^+(q, z) &= -\omega_r^-(q, z) - \frac{1}{2}\chi_{p-r}(q, z), \\ \omega_{r;1}^-(q, z) &= -\omega_r^+(q, z) + \frac{1}{2}\chi_r(q, z) \end{aligned}$$

for $1 \leq r \leq p-1$, and $\omega_{p;1}^\pm(q, z) = \omega_p^\pm(q, z)$.

C.2. Modular transformation properties. Under the modular group action, the functions $\chi_r^\pm(\tau, \nu)$ and $\omega_r^\pm(\tau, \nu)$, $1 \leq r \leq p$, and $\chi_r(\tau, \nu)$, $1 \leq r \leq p-1$, generate a representation whose structure can be described as a deformation of the representation

$$(C.6) \quad \mathcal{R}_{p+1} \oplus \mathbb{C}^2 \otimes \mathcal{R}_{p+1} \oplus \mathcal{R}_{p-1}^{\text{int}} \oplus \mathbb{C}^2 \otimes \mathcal{R}_{p-1}^{\text{int}} \oplus \mathbb{C}^3 \otimes \mathcal{R}_{p-1}^{\text{int}},$$

where $\mathcal{R}_{p-1}^{\text{int}}$ is the $(p-1)$ -dimensional $SL(2, \mathbb{Z})$ representation on the integrable $\widehat{sl}(2)_k$ characters, \mathcal{R}_{p+1} is a $(p+1)$ -dimensional representation, \mathbb{C}^2 is the defining two-dimensional representation, and \mathbb{C}^3 is its symmetrized square. The deformation amounts to the occurrence of “lower” representation characters in the right-hand side of transformations of the “higher” ones.

C.2.1. The lowest in this sense are the integrable $\widehat{sl}(2)$ -representation characters χ_r , $1 \leq r \leq p-1$, which span the representation $\mathcal{R}_{p-1}^{\text{int}}$:

$$(C.7) \quad \chi_r(\tau+1, \nu) = \lambda_{r,p} \chi_r(\tau, \nu), \quad \lambda_{r,p} = e^{i\pi(\frac{r^2}{2p} - \frac{1}{4})},$$

$$(C.8) \quad \chi_r(-\frac{1}{\tau}, \frac{\nu}{\tau}) = \sqrt{\frac{2}{p}} e^{i\pi k \frac{\nu^2}{2\tau}} \sum_{s=1}^{p-1} \sin \frac{\pi r s}{p} \chi_s(\tau, \nu),$$

C.2.2. Next comes the representation \mathcal{R}_{p+1} spanned by the linear combinations

$$(C.9) \quad \begin{aligned} \pi_0(\tau, \nu) &= \omega_p^-(\tau, \nu), \\ \pi_r(\tau, \nu) &= \omega_r^+(\tau, \nu) + \omega_{p-r}^-(\tau, \nu), \quad 1 \leq r \leq p-1, \\ \pi_p(\tau, \nu) &= \omega_p^+(\tau, \nu), \end{aligned}$$

which transform as

$$(C.10) \quad \begin{aligned} \pi_r(\tau+1, \nu) &= \lambda_{r,p} \pi_r(\tau, \nu), \\ \pi_r(-\frac{1}{\tau}, \frac{\nu}{\tau}) &= i \sqrt{\frac{2}{p}} e^{i\pi k \frac{\nu^2}{2\tau}} \left(\frac{1}{2} \pi_0(\tau, \nu) + \frac{(-1)^r}{2} \pi_p(\tau, \nu) + \sum_{s=1}^{p-1} \cos \frac{\pi rs}{p} \pi_s(\tau, \nu) \right) \end{aligned}$$

for $0 \leq r \leq p$. Next, (a deformation of) the $\mathbb{C}^2 \otimes \mathcal{R}_{p-1}^{\text{int}}$ representation is spanned by the linear combinations

$$(C.11) \quad \begin{aligned} \overline{\omega}_r(\tau, \nu) &= (p-r) \omega_r^+(\tau, \nu) - r \omega_{p-r}^-(\tau, \nu), \\ \varsigma_r(\tau, \nu) &= \tau \overline{\omega}_r(\tau, \nu), \end{aligned} \quad 1 \leq r \leq p-1,$$

which transform as

$$(C.12) \quad \begin{aligned} \overline{\omega}_r(\tau+1, \nu) &= \lambda_{r,p} \overline{\omega}_r(\tau, \nu), \quad \varsigma_r(\tau+1, \nu) = \lambda_{r,p} (\varsigma_r(\tau, \nu) + \overline{\omega}_r(\tau, \nu)), \\ \overline{\omega}_r(-\frac{1}{\tau}, \frac{\nu}{\tau}) &= \sqrt{\frac{2}{p}} e^{i\pi k \frac{\nu^2}{2\tau}} \sum_{s=1}^{p-1} \sin \frac{\pi rs}{p} \left(\varsigma_s(\tau, \nu) - \frac{p\nu}{2} \chi_s(\tau, \nu) \right), \\ \varsigma_r(-\frac{1}{\tau}, \frac{\nu}{\tau}) &= \sqrt{\frac{2}{p}} e^{i\pi k \frac{\nu^2}{2\tau}} \sum_{s=1}^{p-1} \sin \frac{\pi rs}{p} \left(-\overline{\omega}_s(\tau, \nu) + \frac{p\nu}{2\tau} \chi_s(\tau, \nu) \right) \end{aligned}$$

(the deformation is due to ν times the integrable-representation characters occurring in the right-hand side).

C.2.3. Further, the linear combinations of the $W(k)$ -characters

$$\begin{aligned} \rho_0(\tau, \nu) &= \chi_p^-(\tau, \nu), \\ \rho_r(\tau, \nu) &= \chi_r^+(\tau, \nu) + \chi_{p-r}^-(\tau, \nu) + \frac{r}{2p} \chi_r(\tau, \nu), \quad 1 \leq r \leq p-1, \\ \rho_p(\tau, \nu) &= \chi_p^+(\tau, \nu) \end{aligned}$$

transform as

$$\begin{aligned} \rho_r(\tau+1, \nu) &= \lambda_{r,p} \rho_r(\tau, \nu), \\ \rho_r(-\frac{1}{\tau}, \frac{\nu}{\tau}) &= i \sqrt{\frac{2}{p}} e^{i\pi k \frac{\nu^2}{2\tau}} \left(\frac{1}{2} (\tau \rho_0(\tau, \nu) + \nu \pi_0(\tau, \nu)) + \frac{(-1)^r}{2} (\tau \rho_p(\tau, \nu) + \nu \pi_p(\tau, \nu)) \right. \\ &\quad \left. + \sum_{s=1}^{p-1} \cos \frac{\pi rs}{p} (\tau \rho_s(\tau, \nu) + \nu \pi_s(\tau, \nu)) \right). \end{aligned}$$

Here, $\tau \rho_r(\tau, \nu)$ are to be regarded as new functions, with the modular transformations for them to be (easily) obtained from the above formulas (for example, $\tau \rho_r \mapsto \lambda_{r,p} \tau \rho_r + \lambda_{r,p} \rho_r$

under $\tau \mapsto \tau + 1$; we do not introduce a special notation for $\tau\rho_r$). Modulo the v -terms in the right-hand sides, $(\rho_r, \tau\rho_r)$ then span the $SL(2, \mathbb{Z})$ representation $\mathbb{C}^2 \otimes \mathcal{R}_{p+1}$.

Finally, the linear combinations of the characters

$$\varphi_r(\tau, v) = (p-r)\chi_r^+(\tau, v) - r\chi_{p-r}^-(\tau, v) - \left(\frac{r^2}{4p} + \frac{1}{8i\pi\tau}\right)\chi_r(\tau, v), \quad 1 \leq r \leq p-1,$$

transform as

$$\begin{aligned} \varphi_r(\tau+1, v) &= \lambda_{r,p}\varphi_r(\tau, v), \\ \varphi_r\left(-\frac{1}{\tau}, \frac{v}{\tau}\right) &= \sqrt{\frac{2}{p}} e^{i\pi k \frac{v^2}{2\tau}} \sum_{s=1}^{p-1} \sin \frac{\pi rs}{p} \left(\tau^2 \varphi_s(\tau, v) + v\tau \varpi_s(\tau, v) - \frac{pv^2}{4} \chi_s(\tau, v) \right). \end{aligned}$$

Here, too, $(\varphi_r, \tau\varphi_r, \tau^2\varphi_r)$ form the triplet $\mathbb{C}^3 \otimes \mathcal{R}_{p-1}^{\text{int}}$ modulo the explicitly v -dependent terms.

APPENDIX D. ABC IDENTITIES

We here derive the “open periodicity” and some other symmetries of the higher string functions.

D.1. Lemma. *For*

$$(D.1) \quad C_{n,r}(q) = \sum_{a \in \mathbb{Z}} \sum_{j \geq 1} (-1)^{j+1} \left(q^{\frac{1}{2}j(j-n)+ra+pa^2+\frac{1}{2}j(2ap+r)} - (r \mapsto -r) \right),$$

$$(D.2) \quad B_{n,r}(q) = \sum_{a \in \mathbb{Z}} \sum_{j \geq 1} (-1)^{j+1} a \left(q^{\frac{1}{2}j(j-n)+ra+pa^2+\frac{1}{2}j(2ap+r)} - (r \mapsto -r) \right),$$

$$(D.3) \quad A_{n,r}(q) = \sum_{a \in \mathbb{Z}} \sum_{j \geq 1} (-1)^{j+1} a^2 \left(q^{\frac{1}{2}j(j-n)+ra+pa^2+\frac{1}{2}j(2ap+r)} - (r \mapsto -r) \right),$$

we have the “open quasiperiodicity” formulas

$$\begin{aligned} C_{n+2k\ell,r}(q) &= q^{k\ell^2+n\ell} C_{n,r}(q), \\ B_{n+2k\ell,r}(q) &= q^{k\ell^2+n\ell} B_{n,r}(q) + \ell q^{k\ell^2+n\ell} C_{n,r}(q) + \\ &\quad + \begin{cases} -q^{k\ell^2+n\ell} \sum_{j=1}^{2\ell} (-1)^j q^{-\frac{k}{4}j^2-\frac{n}{2}j-\frac{r^2}{4p}} \eta(q) \bar{\psi}_r^j(q), & \ell \geq 1, \\ q^{k\ell^2+n\ell} \sum_{j=2\ell+1}^0 (-1)^j q^{-\frac{k}{4}j^2-\frac{n}{2}j-\frac{r^2}{4p}} \eta(q) q \bar{\psi}_r^j(q), & \ell \leq -1, \end{cases} \end{aligned}$$

and

$$A_{n+2k\ell,r}(q) = q^{k\ell^2+n\ell} A_{n,r}(q) + 2\ell q^{k\ell^2+n\ell} B_{n,r}(q) + \ell^2 q^{k\ell^2+n\ell} C_{n,r}(q) +$$

$$+ \begin{cases} -q^{k\ell^2+n\ell} \sum_{j=1}^{2\ell} (2\ell-j)(-1)^j q^{-\frac{k}{4}j^2 - \frac{n}{2}j - \frac{r^2}{4p}} \eta(q) \psi_r^{\bar{j}}(q), & \ell \geq 1, \\ q^{k\ell^2+n\ell} \sum_{j=2\ell+1}^0 (2\ell-j)(-1)^j q^{-\frac{k}{4}j^2 - \frac{n}{2}j - \frac{r^2}{4p}} \eta(q) \psi_r^{\bar{j}}(q), & \ell \leq -1, \end{cases}$$

where

$$(D.4) \quad \psi_r^{\bar{j}}(q) = \begin{cases} \psi_q^+(q), & j \text{ even}, \\ \psi_q^-(q), & j \text{ odd}. \end{cases}$$

Definition (D.4) is an excusable abuse of notation. The formula for $C_{n+2k\ell,r}(q)$ is of course the classic string-function “quasiperiodicity.”

Proof. The properties claimed in the lemma are particular cases of a general formula derived as follows. For a (polynomial) function f defined on \mathbb{Z} , we set

$$F_{n,r}(q) = \sum_{a \in \mathbb{Z}} \sum_{j \geq 1} (-1)^{j+1} f(a) \left(q^{\frac{1}{2}j(j-n)+ra+pa^2+\frac{1}{2}j(2ap+r)} - (r \mapsto -r) \right)$$

and then calculate $F_{n+2k\ell,r}$ for any $\ell \in \mathbb{Z}$ by shifting the summation variables as $a \mapsto a + \ell$ and $j \mapsto j - 2\ell$. An elementary calculation then gives

$$\begin{aligned} F_{n+2k\ell,r}(q) &= q^{k\ell^2+n\ell} \sum_{a \in \mathbb{Z}} \sum_{j \geq 2\ell+1} (-1)^{j+1} f(a+\ell) \left(q^{\frac{1}{2}j(j-n)+ra+pa^2+\frac{1}{2}j(2ap+r)} - (r \mapsto -r) \right) \\ &= q^{k\ell^2+n\ell} F_{n,r} \\ &\quad + q^{k\ell^2+n\ell} \sum_{a \in \mathbb{Z}} \sum_{j \geq 1} (-1)^{j+1} (f(a+\ell) - f(a)) \left(q^{\frac{1}{2}j(j-n)+ra+pa^2+\frac{1}{2}j(2ap+r)} - (r \mapsto -r) \right) \\ &\quad + q^{k\ell^2+n\ell} \sum_{a \in \mathbb{Z}} \left[\begin{array}{c} -\sum_{j=1}^{2\ell} \\ 0 \\ \sum_{j=2\ell+1} \end{array} \right] (-1)^{j+1} f(a+\ell) \left(q^{\frac{1}{2}j(j-n)+ra+pa^2+\frac{1}{2}j(2ap+r)} - (r \mapsto -r) \right), \end{aligned}$$

where $-\sum_{j=1}^{2\ell}$ is to be taken for $\ell > 0$ and $\sum_{j=2\ell+1}^0$ for $\ell < 0$. In either of these *finite* sums, we can change the order of summation and then shift the a summation variable to obtain

$$\begin{aligned} F_{n+2k\ell,r}(q) &= q^{k\ell^2+n\ell} \left(F_{n,r} + \right. \\ &\quad + \sum_{a \in \mathbb{Z}} \sum_{j \geq 1} (-1)^{j+1} (f(a+\ell) - f(a)) \left(q^{\frac{1}{2}j(j-n)+ra+pa^2+\frac{1}{2}j(2ap+r)} - (r \mapsto -r) \right) \\ &\quad + \left. \left[\begin{array}{c} -\sum_{j=1}^{2\ell} \\ 0 \\ \sum_{j=2\ell+1} \end{array} \right] (-1)^{j+1} q^{-\frac{k}{4}j^2 - \frac{n}{2}j} \sum_{a \in \mathbb{Z} + \frac{j}{2}} (f(\ell - \frac{j}{2} + a) - f(\ell - \frac{j}{2} - a)) q^{pa^2+ra} \right). \end{aligned}$$

For $f(a) = 1$, a , and a^2 , the respective “ F ”-functions are $C_{n,r}(q)$, $B_{n,r}(q)$, and $A_{n,r}(q)$, with the results stated in the lemma. \square

D.1.1. For $\mathcal{C}_{n,r}(q)$, $\mathcal{B}_{n,r}(q)$, and $\mathcal{A}_{n,r}(q)$ expressed as in (2.8), the formulas in **D.1** are restated as follows: first, $\mathcal{C}_{n+2k\ell,r}(q) = \mathcal{C}_{n,r}(q)$, and then

$$\mathcal{B}_{n+2k\ell,r}(q) = \mathcal{B}_{n,r}(q) + \begin{cases} -\sum_{j=1}^{2\ell} (-1)^j q^{-\frac{k}{4}(j+\frac{n}{k})^2} \frac{\bar{\psi}_r^j(q)}{\eta(q)}, & \ell \geq 1, \\ \sum_{j=2\ell+1}^0 (-1)^j q^{-\frac{k}{4}(j+\frac{n}{k})^2} \frac{\bar{\psi}_r^j(q)}{\eta(q)}, & \ell \leq -1, \end{cases}$$

and

$$\mathcal{A}_{n+2k\ell,r}(q) = \mathcal{A}_{n,r}(q) + \begin{cases} \sum_{j=1}^{2\ell} (j+\frac{n}{k}) (-1)^j q^{-\frac{k}{4}(j+\frac{n}{k})^2} \frac{\bar{\psi}_r^j(q)}{\eta(q)}, & \ell \geq 1, \\ -\sum_{j=2\ell+1}^0 (j+\frac{n}{k}) (-1)^j q^{-\frac{k}{4}(j+\frac{n}{k})^2} \frac{\bar{\psi}_r^j(q)}{\eta(q)}, & \ell \leq -1 \end{cases}$$

(we recall that $\bar{\psi}_r^j(q)$ is defined in (D.4)).

D.2. Lemma. Relations (1.4), (1.14), (1.12), and (1.17) hold.

Proof. The reflection symmetries, Eqs. (1.4) and (1.14), are shown by elementary manipulations with the same $F_{n,r}(q)$ as in **D.1**, which yield

$$\begin{aligned} F_{-n,r}(q) &= \sum_{a \in \mathbb{Z}} \sum_{j \geq 1} (-1)^{j+1} f(-a) \left(q^{\frac{1}{2}j(j-n)+ra+pa^2+\frac{1}{2}j(2ap+r)} - (r \mapsto -r) \right) \\ &\quad + \sum_{a \in \mathbb{Z}} (f(a) - f(-a)) q^{ra+pa^2} \end{aligned}$$

(identity (2.6) was used here in particular). For $f(a) = 1$, we recover the well-known symmetry $C_{-n,r}(q) = C_{n,r}(q)$ and, evidently, $\mathcal{C}_{-n,r}(q) = \mathcal{C}_{n,r}(q)$ for $\mathcal{C}_{n,r}(q)$ defined in (1.3); Eqs. (1.4) and (1.14) also follow immediately.

Next, the first line in (1.12) follows from **D.1.1** and (B.8), and the second line from (1.4), (B.9), and (B.8). Similarly, the first line in (1.17) follows from **D.1.1** and (B.8), and the second line from (1.14), (B.8), and the identity

$$\Phi'(\tau, -\mu) = \Phi'(\tau, \mu) - \frac{\mu}{\tau} e^{-i\pi \frac{\mu^2}{\tau}}$$

obtained by differentiating (B.9) (see (1.15)). \square

D.2.1. With relations (1.4) thus established, it readily follows from **D.1.1** that modulo $\mathbb{C}[\psi_s^\pm(q) q^{-\frac{n^2}{4k}} / \eta(q)]$, the independent $\mathcal{B}_{n,r}(q)$ are $\mathcal{B}_{m,r}(q)$, $1 \leq m \leq k-1$, $1 \leq r \leq p$. In particular, it is easy to see that

$$(D.5) \quad \mathcal{B}_{-k,r}(q) = -\frac{\psi_r^-(q)}{2\eta(q)}, \quad \mathcal{B}_{0,r}(q) = -\frac{\psi_r^+(q)}{2\eta(q)},$$

and so on for $\mathcal{B}_{k\ell,r}(q)$ in accordance with **D.1.1**.

D.2.2. Finally, it is also obvious from the definitions in **1.3** that $\mathcal{C}_{n,0}(q) = \mathcal{B}_{n,0}(q) = \mathcal{A}_{n,0}(q) = 0$. In view of the symmetry

$$(D.6) \quad \mathcal{C}_{n+k,p-r}(q) = \mathcal{C}_{n,r}(q),$$

this also implies that $\mathcal{C}_{n,p}(q) = 0$.

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